

Regularity of quantum τ -functions generated by quantum birational Weyl group actions

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Abstract

We canonically quantize the τ -functions for the birational Weyl group action arising from a nilpotent Poisson algebra proposed by Noumi and Yamada. We also construct the q -difference deformation of the canonical quantization of the τ -functions. Using the translation functors for the symmetrizable Kac-Moody algebras, we prove the regularity of the quantum τ -functions, namely, we show that the quantum τ -functions are polynomials in dependent variables.

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0 Introduction

In the previous paper [16], the author canonically quantized the birational Weyl group action arising from a nilpotent Poisson algebra proposed by Noumi and Yamada [25]. But he did not quantize their τ -functions. In this paper, we shall quantize the τ -functions and prove that the quantized τ -functions are polynomials in dependent variables.

Let $[a_{ij}]_{i,j \in I}$ be any symmetrizable generalized Cartan matrix (GCM for short) with positive integers d_i ($i \in I$) satisfying $d_i a_{ij} = d_j a_{ji}$. Denote by W the Weyl group of the GCM $[a_{ij}]_{i,j \in I}$ generated by the simple reflections s_i ($i \in I$).

0.1 Classical case

Following Noumi and Yamada [25], we define a nilpotent Poisson algebra to be a Poisson commutative integral domain generated by $\{f_i\}_{i \in I}$ as a Poisson algebra with the following nilpotency property of the Poisson bracket:

$$(\text{ad}_{\{\cdot\}} f_i)^{1-a_{ij}}(f_j) = \underbrace{\{f_i, \{\cdots, \{f_i, \{f_i, f_j\}\} \cdots\}}_{1-a_{ij} \text{ times}} = 0 \quad (i \neq j),$$

where $(\text{ad}_{\{\cdot\}} f)(g) = \{f, g\}$. We call f_i 's *the dependent variables*.

In Theorem 1.1 of [25], introducing the Poisson central *parameter variables* α_i^\vee ($i \in I$), they construct the birational Weyl group action by

$$\begin{aligned} s_i(\alpha_j^\vee) &= \alpha_j^\vee - a_{ij} \alpha_i^\vee, \quad s_i(f_i) = f_i, \\ s_i(f_j) &= \exp(\text{ad}_{\{\cdot\}} \alpha_i^\vee \log f_i)(f_j) = \sum_{k=0}^{-a_{ij}} \frac{(\alpha_i^\vee)^k}{k!} (\text{ad}_{\{\cdot\}} f_i)^k(f_j) f_i^{-1} \quad (i \neq j). \end{aligned} \quad (0.1)$$

These formulas shall be canonically quantized by (1.2), (1.5), and (1.6), respectively.

Moreover, in Theorem 1.2 of [25], they introduce *Laurent τ -monomials* τ^μ for integral weights $\mu \in P$ and extend the birational Weyl group action to the τ -monomials by

$$s_i(\tau^\mu) = f_i^{\langle \alpha_i^\vee, \lambda \rangle} \tau^{s_i(\mu)} = f_i^{\langle \alpha_i^\vee, \lambda \rangle} \tau^{\mu - \langle \alpha_i^\vee, \mu \rangle \alpha_i}, \quad (0.2)$$

where α_i^\vee 's are identified with the simple coroots, α_i 's are the simple roots, and $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between the coroot lattice Q^\vee and the weight lattice P . This action

on the Laurent τ -monomials shall be quantized by (1.4), the appearance of which is same as (0.2).

They deal with the τ -cocycle in [25]. However, for the compatibility with the quantum case, we equivalently introduce *the τ -functions* $\tau_{(w(\mu))}$ for $w(\mu) \in WP_+$ by

$$\tau_{(w(\mu))} = w(\tau^\mu) \quad (w \in W, \mu \in P_+).$$

Note that each $w(\tau^\mu)$ depends only on $w(\mu)$ because $s_i(\tau^\mu) = \tau^\mu$ if $\langle \alpha_i^\vee, \mu \rangle = 0$. For any $w(\mu) \in WP_+$, there exists a unique rational function $\phi_w(\mu)$ of f_i 's and α_i^\vee 's such that $\tau_{(w(\mu))} = \phi_w(\mu)\tau^{w(\mu)}$, where $\phi_w(\mu)$ is called *the τ -cocycle* in [25]. Since the quantized version of $\phi_w(\mu)$ does not commute with the quantized τ -monomials in general, we shall deal with the quantized version of $\tau_{(w(\mu))}$ in the quantum case.

In this paper, for the fundamental weights Λ_i , we call $\tau_i = \tau^{\Lambda_i}$ *the τ -variables*. Although they call only τ_i 's the τ -functions in [25], we call all $\tau_{(w(\mu))}$ ($w \in W, \mu \in P_+$) the τ -functions. One should not be confused by the difference of terminologies.

The main result of [25] is the regularity of $\phi_w(\mu)$ for any dominant integral weight $\mu \in P_+$ (Theorem 1.3 of [25]). In other words, they prove that, for any $\mu \in P_+$ and any $w \in W$, the τ -function $\tau_{w(\mu)}$ is a polynomial in f_i 's and α_i^\vee 's.

In [32], one of the author of [25] finds the determinant formulas of the τ -functions for the birational Weyl group actions of type $A_{n-1}^{(1)}$ and A_∞ , and interprets them as Plücker coordinates of the universal Grassmann manifolds in the Sato theory of soliton equations [30]. The determinant formulas immediately lead to the regularity of the τ -functions of type A. In [25], they generalize the Sato theoretic interpretation of the A-type τ -functions to the case for any symmetrizable GCM and show the regularity of the τ -functions for any type.

The regularity of the τ -functions proved by Noumi and Yamada [32], [25] has many corollaries which state polynomiality of certain special rational functions. In particular, polynomialities of rational functions, which give special solutions of the bilinear forms of the Painlevé equations and are generated by the Bäcklund transformations, are corollaries of the regularity of the τ -functions for the birational Weyl group actions.

For example, let $Q_m(x)$ ($m \in \mathbb{Z}_{\geq 0}$) be the rational functions defined by the following recurrence equation:

$$Q_{m-1}Q_{m+1} = Q_m''Q_m - (Q_m')^2 + (x^2 + 2m - 1)Q_m^2, \quad Q_0 = Q_1 = 1. \quad (0.3)$$

In [27], using the analysis of the Painlevé equations in [26], Okamoto proves that all $Q_m(x)$ are polynomials in x (Proposition 5.6 of [27]). The polynomials $Q_m(x)$ are called the Okamoto polynomials. The polynomiality of $Q_m(x)$ is a corollary of the regularity of the τ -functions for the birational Weyl group action of type $A_2^{(1)}$ (Theorem 4.3 of [24]). It is non-trivial to show that the right-hand side of the recurrence equation (0.3) is divisible by $Q_{m-1}(x)$. The original proof of Okamoto is not purely algebraic. On the other hand, the regularity of τ -functions for the birational Weyl group action has a purely algebraic proof. For other examples of special polynomials for the Painlevé equations, see also [33] and references therein.

0.2 Quantization

In this paper, we shall introduce the quantum τ -functions (Section 2.1) and prove their regularity. But the method to prove the regularity is completely different from the one in the classical case of [25]. In order to prove the regularity of the quantum τ -functions, we shall use the translation functors in the representation theory (Section 2.5).

The regularity in the quantized case implies the regularity in the classical case through the classical limit. Therefore we obtain another purely algebraic proof of the regularity of the classical τ -functions. In particular, the polynomiality of the special rational functions for the Painlevé equations generated by the Bäcklund transformations can be derived from the theory of the translation functors for the Kac-Moody algebras. This could be a surprising relationship.

We summarize the implications as below:

$$\begin{aligned} & \exists \text{ exact functor } T(M) \subset M \otimes L(\mu) \text{ with } T(M(w \circ \lambda)) = M(w \circ (\lambda + \mu)) \\ & \implies \text{regularity of the quantum } \tau\text{-functions} \\ & \implies \text{regularity of the classical } \tau\text{-functions} \\ & \implies \text{polynomiality of special rational solutions of the Painlevé equations.} \end{aligned}$$

Here we denote by $w \circ \lambda$ the shifted action of $w \in W$ on $\lambda \in P_+$, by $M(w \circ \lambda)$ the Verma module with highest weight $w \circ \lambda$, by $L(\mu)$ the simple quotient of $M(\mu)$ for $\mu \in P_+$, and by $T = T_\lambda^{\lambda+\mu}$ the translation functor.

The major difficulty of quantizing the τ -functions was the fact that we did not have a natural Poisson algebra which contains the τ -variables τ_i (or the τ -monomials τ^μ). Roughly speaking, “canonical quantization” stands for replacement of the Poisson brackets in a Poisson algebra with the commutators in the corresponding non-commutative associative algebra. But we did not have Poisson brackets for the τ -variables. We should find the appropriate commutation relations for the quantum τ -variables under the situation where the Poisson brackets are unknown.

The answer is very simple. The τ -variables τ_i ($i \in I$) are defined to be the exponentials of the canonical conjugate variables of the parameter variables α_i^\vee ($i \in I$). More precisely, we define τ_i by $\tau_i = \exp(\partial/\partial \alpha_i^\vee)$ ($i \in I$). Then we have $\tau_i \tau_j = \tau_j \tau_i$, $\tau_i \alpha_j^\vee = (\alpha_j^\vee + \delta_{ij}) \tau_i$ ($i, j \in I$) and $\tau_i f_j = f_j \tau_i$. The quantum τ -variables are difference operators of the parameter variables. More generally we assume that $\tau^\lambda \tau^\mu = \tau^{\lambda+\mu}$, $\tau^\mu \alpha_j^\vee = (\alpha_j^\vee + \langle \alpha_j^\vee, \mu \rangle) \tau^\mu$, and $\tau^\mu f_j = f_j \tau^\mu$ for integral weights $\mu, \lambda \in P$.

In the classical case, we assume that $\{\tau^\lambda, \tau^\mu\} = 0$, $\{\tau^\mu, \alpha_j^\vee\} = \langle \alpha_j^\vee, \mu \rangle \tau^\mu$, and $\{\tau_i, f_j\} = 0$. Then the formula (0.2) can be derived as follows:

$$s_i(\tau^\mu) = \exp(\text{ad}_{\{, \}} \alpha_i^\vee \log f_i)(\tau^{s_i(\mu)}) = \exp(\langle \alpha_i^\vee, \mu \rangle \log f_i) \tau^{s_i(\mu)} = f_i^{\langle \alpha_i^\vee, \mu \rangle} \tau^{s_i(\mu)}. \quad (0.4)$$

See also (0.1). Therefore the birational action of s_i is uniformly written in the form $s_i(a) = \exp(\text{ad}_{\alpha_i^\vee} \log f_i)(\tilde{s}_i(a))$, where \tilde{s}_i stands for the Weyl group action on the parameter variables (or coroots) and the integral weights which trivially acts on f_i 's.

In the quantum case, we shall construct the quantum birational Weyl group action by $s_i(a) = f_i^{\alpha_i^\vee} \tilde{s}_i(a) f_i^{-\alpha_i^\vee}$, where f_i 's are the generators of the associative algebra the fundamen-

tal relations of which are the Serre (or q -Serre) relations, and are called the (quantum) dependent variables. (For details, see Section 1.2.) This is an almost straightforward canonical quantization of the classical birational Weyl group action $s_i(a) = \exp(\text{ad } \alpha_i^\vee \log f_i)(\tilde{s}_i(a))$.

The fractional powers $f_i^{\alpha_i^\vee}$ shall be constructed in Section 1.5. The parameter variables are identified with the simple coroots also in the quantum case. For $\lambda \in P$ and a function a of the parameter variables α_i^\vee ($i \in I$) which contains the fractional powers of f_i ($i \in I$), we denote by $\phi_\lambda(a)$ the value of a at λ , which is obtained by the substitution of $\langle \alpha_i^\vee, \lambda \rangle$ into α_i^\vee . We shall define the fractional powers so that $a = 0$ if and only if $\phi_\lambda(a) = 0$ for all integral weights $\lambda \in P$. For any rational function a of f_i 's and α_i^\vee 's, it is sufficient for $a = 0$ that $\phi_{\lambda+\rho}(a) = 0$ for all dominant integral weights $\lambda \in P_+$, where ρ denotes the Weyl vector $\sum_{i \in I} \Lambda_i$. Therefore the calculations of the quantum birational Weyl group action reduces to those of $f_i^{\langle \alpha_i^\vee, \lambda+\rho \rangle} \phi_{\lambda+\rho}(a) f_i^{-\langle \alpha_i^\vee, \lambda+\rho \rangle}$ for dominant integral weights $\lambda \in P_+$.

For the proof of the regularity of the quantum τ -functions, we can assume that f_i 's generate the lower triangular part of $U(\mathfrak{g})$ or $U_q(\mathfrak{g})$, where \mathfrak{g} is the Kac-Moody Lie algebra of type $[a_{ij}]_{i,j \in I}$. Let $M(\lambda)$ be the Verma module over $U(\mathfrak{g})$ or $U_q(\mathfrak{g})$ with highest weight $\lambda \in P$. Denote by v_λ a highest weight vector of $M(\lambda)$. Then, for each $\lambda \in P$ with $\langle \alpha_i^\vee, \lambda \rangle \geq 0$, the vector $f_i^{\langle \alpha_i^\vee, \lambda+\rho \rangle} v_\lambda$ is a singular vector with weight $s_i \circ \lambda = s_i(\lambda + \rho) - \rho$. In this way, we can relate the quantum birational Weyl group action with the singular vectors in the Verma modules with dominant integral highest weights. For details, see Section 2.4.

Consequently, we can reduce the regularity of the quantum τ -functions to the divisibility (from the right) of the singular vectors in $M(\lambda + \mu)$ by the corresponding singular vectors in $M(\lambda)$ for any $\lambda, \mu \in P_+$ (Proposition 2.9). The translation functor $T_\lambda^{\lambda+\mu}$ for the symmetrizable Kac-Moody algebra \mathfrak{g} makes a connection between the singular vectors in $M(\lambda)$ and those in $M(\lambda + \mu)$ and then proves the divisibility of the singular vectors in the Kac-Moody case (Section 2.5). The equivalence (shown by Etingof and Kazhdan in [6]) between the category \mathcal{O} for \mathfrak{g} and that of $U_q(\mathfrak{g})$ shows that the divisibility of the singular vectors in the Verma modules over $U_q(\mathfrak{g})$ also reduces to the theory of the translation functor for the symmetrizable Kac-Moody algebra (Section 2.6). Thus the regularity of the quantum τ -functions is proved both in the Kac-Moody case and in the q -difference case. This is the main result of this paper.

0.3 Conventions

We adopt the following conventions.

The term “quantization” stands for “canonical quantization”, not for “ q -difference deformation”. For example, “the quantum q -Hirota-Miwa equation” does not mean “the q -difference deformation of the Hirota-Miwa equation” but “the q -difference deformation of the canonically quantized Hirota-Miwa equation”. (See Section 2.2 for details.)

We shall deal with both canonical quantizations and their q -difference analogues. For example, for a symmetrizable Kac-Moody Lie algebra \mathfrak{g} , the universal enveloping algebra $U(\mathfrak{g})$ can be regarded as the canonical quantization of the symmetric algebra $S(\mathfrak{g})$, and the q -difference deformation of $U(\mathfrak{g})$ is denoted by $U_q(\mathfrak{g})$. We shall construct the quantum τ -functions both for $U(\mathfrak{g})$ and for $U_q(\mathfrak{g})$.

A field is always commutative. A skew field stands for a possibly non-commutative field. An associative algebra over a field shall be always with the unit 1. Denote the set of all non-negative integers by $\mathbb{Z}_{\geq 0}$.

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1 Quantization of birational Weyl group actions

In this section, we extend the quantized birational Weyl group action on the dependent variables f_i and the parameter variables α_i^\vee (Theorem 4.3 of [16]) to the τ -variables τ_i .

1.1 Symmetrizable GCM and Weyl group

Throughout this paper, a matrix $[a_{ij}]_{i,j \in I}$ stands for a symmetrizable generalized Cartan matrix (GCM for short) symmetrized by positive integers $\{d_i\}_{i \in I}$. In other words, we assume that $[a_{ij}]_{i,j \in I}$ is an integer matrix with

$$a_{ii} = 2; \quad a_{ij} \leq 0 \text{ if } i \neq j; \quad a_{ij} = 0 \text{ if and only if } a_{ji} = 0; \quad d_i a_{ij} = d_j a_{ji}.$$

Let Q^\vee be a free \mathbb{Z} -module and set $P = \text{Hom}(Q^\vee, \mathbb{Z})$. Denote by $\langle \cdot, \cdot \rangle : Q^\vee \times P \rightarrow \mathbb{Z}$ the canonical pairing. Assume that $\alpha_i^\vee \in Q^\vee$ and $\alpha_i \in P$ ($i \in I$) satisfy the following conditions: (1) $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ ($i, j \in I$); (2) $\{\alpha_i^\vee\}_{i \in I}$ is linearly independent over \mathbb{Z} ; (3) $\alpha_i \neq \alpha_j$ if $i \neq j$. We call Q^\vee the coroot lattice, P the weight lattice, α_i^\vee 's the simple coroots, and α_i 's the simple roots. We set $P_+ = \{\lambda \in P \mid \langle \alpha_i^\vee, \lambda \rangle \geq 0 \text{ } (i \in I)\}$, $Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$, and $Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. We call an element of P_+ a dominant integral weight and Q the root lattice. Assume that weights $\Lambda_j \in P_+$ ($j \in I$) satisfy $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$ ($i, j \in I$). We call Λ_j 's the fundamental weights. The Weyl vector ρ is defined by $\rho = \sum_{i \in I} \Lambda_i$.

Let W be the Weyl group of the GCM $[a_{ij}]_{i,j \in I}$, namely W is defined to be the group generated by $\{s_i\}_{i \in I}$ with following fundamental relations: $s_i s_j = s_j s_i$ if $(a_{ij}, a_{ji}) = (0, 0)$; $s_i s_j s_i = s_j s_i s_j$ if $(a_{ij}, a_{ji}) = (-1, -1)$; $(s_i s_j)^2 = (s_j s_i)^2$ if $(a_{ij}, a_{ji}) = (-1, -2)$; $(s_i s_j)^3 = (s_j s_i)^3$ if $(a_{ij}, a_{ji}) = (-1, -3)$; $s_i^2 = 1$. The Weyl group W acts on Q^\vee and P by

$$s_i(\beta) = \beta - \langle \beta, \alpha_i \rangle \alpha_i^\vee, \quad s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i.$$

In particular, we have $s_i(\alpha_j^\vee) = \alpha_j^\vee - a_{ji} \alpha_i^\vee$ and $s_i(\Lambda_j) = \Lambda_j - \delta_{ij} \alpha_i$. The Weyl group action preserves the canonical pairing $\langle \cdot, \cdot \rangle$ between Q^\vee and P :

$$\langle w(\beta), w(\lambda) \rangle = \langle \beta, \lambda \rangle \quad \text{for } \beta \in Q^\vee, \lambda \in P, w \in W.$$

1.2 Quantum algebras of dependent variables

Let \mathfrak{h} be the vector space over \mathbb{C} isomorphic to $Q_\mathbb{C}^\vee = Q^\vee \otimes \mathbb{C}$. Denote by h_i (resp. h_β) the element of \mathfrak{h} corresponding to $\alpha_i^\vee \otimes 1 \in Q_\mathbb{C}^\vee$ (resp. $\beta \in Q_\mathbb{C}^\vee$). Note that we shall not identify h_β with $\beta \in Q_\mathbb{C}^\vee$.

Let \mathfrak{g} be the Kac-Moody Lie algebra of type $[a_{ij}]_{i,j \in I}$ over \mathbb{C} , namely \mathfrak{g} is defined to be the Lie algebra over \mathbb{C} generated by $\{e_i, f_i, h \mid i \in I, h \in \mathfrak{h}\}$ with fundamental relations:

$$\begin{aligned} [h_\beta, h_\gamma] &= 0, \quad h_\beta + h_\gamma = h_{\beta+\gamma} \quad (\beta, \gamma \in Q_{\mathbb{C}}^\vee), \\ [h_\beta, e_j] &= \langle \beta, \alpha_j \rangle e_j, \quad [h_\beta, f_j] = -\langle \beta, \alpha_j \rangle f_j \quad (\beta \in Q_{\mathbb{C}}^\vee), \quad [e_i, f_j] = \delta_{ij} h_i, \\ \sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} &= 0, \quad \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0 \quad \text{if } i \neq j, \end{aligned}$$

where $e_i^{(k)} = e_i/k!$ and $f_i^{(k)} = f_i/k!$. The last two relations are called the Serre relations. Let \mathfrak{n}_- (resp. \mathfrak{n}_+) be the Lie subalgebra of \mathfrak{g} generated by $\{f_i\}_{i \in I}$ (resp. by $\{e_i\}_{i \in I}$). Denote by $U(\mathfrak{a})$ the universal enveloping algebra of a Lie algebra \mathfrak{a} .

Note that Q^\vee is not regarded as a subset of \mathfrak{h} . We shall assume that elements of Q^\vee commute with f_i 's.

In order to deal with q -difference analogues, we introduce the q -numbers, q -factorials, and the q -binomial coefficients by

$$\begin{aligned} [a]_q &= \frac{q^a - q^{-a}}{q - q^{-1}}, \quad [k]_q! = [1]_q [2]_q \cdots [k]_q, \\ \begin{bmatrix} a \\ k \end{bmatrix}_q &= \frac{[a]_q [a-1]_q [a-2]_q \cdots [a-k+1]_q}{[k]_q!} \quad (k \in \mathbb{Z}_{\geq 0}). \end{aligned}$$

Put $q_i = q^{d_i}$ for $i \in I$.

Let $U_q(\mathfrak{g})$ be the q -difference analogue of $U(\mathfrak{g})$, namely $U_q(\mathfrak{g})$ is defined to be the associative algebra over $\mathbb{C}(q)$ generated by $\{e_i, f_i, q^\beta \mid i \in I, \beta \in Q^\vee\}$ with fundamental relations:

$$\begin{aligned} q^\beta q^\gamma &= q^{\beta+\gamma}, \quad q^0 = 1 \quad (\beta, \gamma, 0 \in Q^\vee), \\ q^\beta e_j q^{-\beta} &= q^{\langle \beta, \alpha_j \rangle} e_j, \quad q^\beta f_j q^{-\beta} = q^{-\langle \beta, \alpha_j \rangle} f_j \quad (\beta \in Q^\vee), \quad [e_i, f_j] = \delta_{ij} [h_i]_{q_i}, \\ \sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} &= 0, \quad \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0 \quad \text{if } i \neq j, \end{aligned}$$

where $e_i^{(k)} = e_i/[k]_{q_i}!$ and $f_i^{(k)} = f_i/[k]_{q_i}!$. The last two relations are called the q -Serre relations. Let $U_q(\mathfrak{n}_-)$ (resp. $U_q(\mathfrak{n}_+)$) be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{f_i\}_{i \in I}$ (resp. $\{e_i\}_{i \in I}$).

The case where we deal with the Kac-Moody Lie algebra \mathfrak{g} (resp. the q -difference deformation $U_q(\mathfrak{g})$ of $U(\mathfrak{g})$) shall be called *the Kac-Moody case* (resp. *the q -difference case*).

In the Kac-Moody (resp. q -difference) case, let A be a residue class algebra of $U(\mathfrak{n}_-)$ (resp. $U_q(\mathfrak{n}_-)$) and assume that A is an integral domain, namely an associative algebra without non-zero zero divisors. Denote the images of f_i 's in A by the same symbols. We also assume that $f_i \neq 0$ in A for all $i \in I$. We call f_i 's *the dependent variables*. We shall construct the quantum birational Weyl group action on certain extensions of A .

We denote by $Q(R)$ the skew field of fractions of an Ore domain R . Any element of $Q(R)$ can be expressed as as^{-1} and $t^{-1}b$ for some $a, b, s, t \in A$ with $s \neq 0$ and $t \neq 0$. We

denote by $F(x_1, \dots, x_n)$ the skew field of fractions of the polynomial ring $F[x_1, \dots, x_n]$ over a skew field F .

If the GCM is of finite or affine type, then A is always an Ore domain (Theorem 2.12 of [16]). When A is an Ore domain, we denote $Q(A)$ by K .

Remark 1.1. We refer [31], [11], [18], [7], Section 3.6 of [3], and Section 2 of [16] for the theory of localization of non-commutative rings.

A Noetherian domain is always an Ore domain (2.1.15 of [18], Corollary 6.7 of [7]). The universal enveloping algebra of a finite dimensional Lie algebra is a Noetherian domain and hence an Ore domain. The q -difference deformation $U_q(\mathfrak{g})$ of $U(\mathfrak{g})$ and its lower triangular part $U_q(\mathfrak{n}_-)$ are always integral domains (Section 7.3 of [13]). If the GCM is of finite type, then $U_q(\mathfrak{g})$ and $U_q(\mathfrak{n}_-)$ are Noetherian (Section 7.4 of [13]) and hence Ore domains.

Let R be an associative algebra over a field with a increasing filtration $R = \bigcup_{k=0}^{\infty} R_k$ such that each R_k is a finite dimensional subspace of R and $1 \in R_0$. We call R the *tempered domain* if R is an integral domain and the convergence radius of $\sum_{k=0}^{\infty} \dim(R_k)z^k$ is not less than 1. A tempered domain is always an Ore domain (Lemma 2.9 of [16], Lemma 1.2 of [29]). This result is very useful for proving that a given algebra is an Ore domain. In particular, it follows that, if the GCM is of finite or affine type, then $U(\mathfrak{g})$, $U_q(\mathfrak{g})$, and their all subquotient domains are Ore domains (Theorem 2.12 of [16]).

Consider the polynomial ring $R = F[x_1, \dots, x_n]$ over a skew field F and assume that $m = 0, 1, 2, \dots, n$ and c_1, \dots, c_m are central elements of F . Then, applying Theorem 2.1 of [31] to R , we obtain that $S = \{f \in R \mid f(c_1, \dots, c_m, x_{m+1}, \dots, x_n) \neq 0\}$ is an Ore set in R . \square

Example 1.2. Assume that $\varepsilon_{ij} \in \{0, \pm 1\}$, $\varepsilon_{ji} = -\varepsilon_{ij}$, and $\varepsilon_{ij} \neq 0$ if and only if $a_{ij} \neq 0$ and $i \neq j$. Set $c_{ij} = \varepsilon_{ij}d_i a_{ij}$ for $i, j \in I$. If A is the associative algebra over \mathbb{C} generated by $\{f_i\}_{i \in I}$ with fundamental relations $[f_i, f_j] = c_{ij}$ ($i, j \in I$), then A is a quotient Ore domain of $U(\mathfrak{n}_-)$. If A is the associative algebra over $\mathbb{C}(q)$ generated by $\{f_i\}_{i \in I}$ with fundamental relations $f_j f_i = q^{c_{ij}} f_i f_j$ ($i, j \in I$), then A is a quotient Ore domain of $U_q(\mathfrak{n}_-)$. In the both cases, we have the skew field $K = Q(A)$ of fractions of A . \square

If R is an associative algebra and S is an Ore set in R , then we have the localization $R[S^{-1}]$ of R with respect to S . For any $c \in R[S^{-1}]$, there exist some $a, b \in R$ and $s, t \in S$ such that $c = as^{-1} = t^{-1}b$.

The multiplicative subset of A generated by $\{f_i\}_{i \in I}$ is an Ore set in A owing to the Serre and q -Serre relations. Therefore we have the localization $\tilde{A} = A[f_i^{-1} \mid i \in I]$ of A with respect to it.

We have $A \subset \tilde{A}$ in any case and $\tilde{A} \subset K = Q(A)$ in the case where A is an Ore domain.

Example 1.3. Assume that the GCM is of type A_2 : $I = \{1, 2\}$, $a_{ii} = 2$, $a_{12} = a_{21} = -1$, $d_i = 1$. Then the Weyl algebra $A = \mathbb{C}[x, \partial]$, where $\partial = d/dx$, can be regarded as a quotient Ore domain of $U(\mathfrak{n}_-)$. The surjective algebra homomorphism from $U(\mathfrak{n}_-)$ onto $A = \mathbb{C}[x, \partial]$ is given by $f_1 \mapsto x$ and $f_2 \mapsto \partial$. Then the multiplicative subset of $A = \mathbb{C}[x, \partial]$ generated by x and ∂ is an Ore set in $A = \mathbb{C}[x, \partial]$ and we obtain $\tilde{A} = \mathbb{C}[x^{\pm 1}, \partial^{\pm 1}]$. For any polynomial

$f \in \mathbb{C}[x]$, we have

$$\partial^{-1}f = \sum_{k=0}^{\infty} (-1)^k f^{(k)} \partial^{-k-1} = f\partial^{-1} - f'\partial^{-2} + f''\partial^{-3} - \dots \quad \text{in } \tilde{A} = \mathbb{C}[x^{\pm 1}, \partial^{\pm 1}].$$

The right-hand side reduces to a finite sum. \square

Example 1.4. The Weyl algebra $A = \mathbb{C}[x, \partial]$, $\partial = d/dx$, can be regarded as a quotient Ore domains of $U(\mathfrak{n}_-)$ of various affine types:

$$\begin{aligned} D_4^{(1)} : \quad & f_2 = \partial, \quad f_i = x - a_i \quad (a_i \in \mathbb{C}, i = 0, 1, 3, 4), \\ B_3^{(1)} : \quad & f_1 = \partial, \quad f_2 = x, \quad f_3 = x - a, \quad f_0 = (x - b)^2 \quad (a, b \in \mathbb{C}), \\ A_3^{(1)} : \quad & f_1 = \partial, \quad f_2 = x, \quad f_3 = x - a, \quad f_0 = \partial - b \quad (a, b \in \mathbb{C}), \\ G_2^{(1)} : \quad & f_1 = \partial, \quad f_2 = x, \quad f_0 = (x - a)^3 \quad (a \in \mathbb{C}), \\ A_2^{(1)} : \quad & f_1 = \partial, \quad f_2 = x, \quad f_0 = \partial + x, \\ D_5^{(2)} : \quad & f_1 = \partial, \quad f_2 = x^2, \quad f_0 = (x - a)^2 \quad (a \in \mathbb{C}), \\ C_2^{(1)} : \quad & f_1 = \partial, \quad f_2 = x^2, \quad f_0 = \partial - a \quad (a \in \mathbb{C}), \\ A_2^{(2)} : \quad & f_1 = \partial, \quad f_0 = x^4, \\ A_1^{(1)} : \quad & f_1 = \partial, \quad f_0 = \partial + x^2. \end{aligned}$$

The GCM's of type $D_4^{(1)}$, $A_3^{(1)}$, and $A_2^{(1)}$ are simply-laced, those of $C_2^{(1)}$ and $A_1^{(1)}$ are symmetric, and those of $B_3^{(1)}$, $G_2^{(1)}$, $D_5^{(2)}$, and $A_2^{(2)}$ are not symmetric but foldings of $D_4^{(1)}$.

For example, in the case of $G_2^{(1)}$, from $f_1 = \partial$, $f_2 = x$, and $f_0 = (x - a)^3$, it follows that $[f_1, [f_1, f_2]] = 0$, $[f_2, [f_2, f_1]] = 0$, $[f_1, [f_1, [f_1, [f_1, f_0]]]] = 0$, $[f_0, [f_0, f_1]] = 0$, and $[f_2, f_0] = 0$. These are the Serre relations of type $G_2^{(1)}$.

Recall that the Painlevé equations P_{VI} , P_V , P_{IV} , P_{III} , and P_{II} have the birational Weyl group symmetries of type $D_4^{(1)}$, $A_3^{(1)}$, $A_2^{(1)}$, $C_2^{(1)}$, and $A_1^{(1)}$, respectively. By Proposition 5.13 of [34], we have the following isomorphisms of the affine Weyl groups:

$$\begin{aligned} W(B_3^{(1)}) &\cong W(A_3^{(1)}) \rtimes \mathbb{Z}/2\mathbb{Z}, & W(G_2^{(1)}) &\cong W(A_2^{(1)}) \rtimes \mathbb{Z}/2\mathbb{Z}, \\ W(D_5^{(2)}) &\cong W(C_2^{(2)}) \rtimes \mathbb{Z}/2\mathbb{Z}, & W(A_2^{(2)}) &\cong W(A_1^{(1)}) \rtimes \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

In fact, the above list is related to the canonical quantization of the Painlevé equations. For the construction of the quantum Painlevé equations, see [12], [22], [20], and [21]. The Weyl group symmetry of the quantum Painlevé equations are naturally interpreted from the perspective of the quantum birational Weyl group actions defined in this paper or in [16]. \square

1.3 Quantum algebras with parameter variables

In the Kac-Moody case, we consider the polynomial rings $A^{\text{pa}} = A[\beta | \beta \in Q^\vee]$ and $\tilde{A}^{\text{pa}} = \tilde{A}[\beta | \beta \in Q^\vee]$ generated by any free basis of the coroot lattice Q^\vee over A and \tilde{A} respectively,

where $\beta \in Q^\vee$'s are central in them:

$$\beta f_j = f_j \beta, \quad \beta \gamma = \gamma \beta \quad (\beta, \gamma \in Q^\vee).$$

We naturally regard the coroot lattice Q^\vee as a subset of A^{pa} and \tilde{A}^{pa} . We call $\beta \in Q^\vee$ *the parameter variables*.

In the q -difference case, we consider the Laurent polynomial rings $A^{\text{pa}} = A[q^\beta | \beta \in Q^\vee]$ and $\tilde{A}^{\text{pa}} = \tilde{A}[q^\beta | \beta \in Q^\vee]$ spanned by $\{q^\beta\}_{\beta \in Q^\vee}$ over A and \tilde{A} respectively, where q^β 's are central in them:

$$q^\beta f_j = f_j q^\beta, \quad q^\beta q^\gamma = q^{\beta+\gamma} \quad (\gamma, \beta \in Q^\vee).$$

We call q^β 's *the parameter variables*.

In the both cases, if A is an Ore domain, then A^{pa} is also an Ore domain. We denote $Q(A^{\text{pa}})$ by K^{pa} shortly, namely, $K^{\text{pa}} = K(\beta | \beta \in Q^\vee)$ in the Kac-Moody case and $K^{\text{pa}} = K(q^\beta | \beta \in Q^\vee)$ in the q -difference case. We have $K^{\text{pa}} = Q(\tilde{A}^{\text{pa}})$.

We call A^{pa} , \tilde{A}^{pa} , and K^{pa} *the quantum algebras with parameter variables*. Also in the following, the symbol “ $()^{\text{pa}}$ ” stands for “*with parameter variables*”.

In the Kac-Moody (resp. q -difference) case, for each $\lambda \in P$, we define the algebra homomorphism $\phi_\lambda : \tilde{A}^{\text{pa}} \rightarrow \tilde{A}$ by

$$\phi_\lambda(a) = a, \quad \phi_\lambda(\beta) = \langle \beta, \lambda \rangle \quad (\text{resp. } \phi_\lambda(q^\beta) = q^{\langle \beta, \lambda \rangle}) \quad (a \in \tilde{A}, \beta \in Q^\vee).$$

This homomorphism ϕ_λ substitutes $\langle \beta, \lambda \rangle$ into $\beta \in Q^\vee$. Thus we obtain the injective algebra homomorphism $\phi : \tilde{A}^{\text{pa}} \rightarrow \tilde{A}^P$ by $\phi(a) = (\phi_\lambda(a))_{\lambda \in P}$ ($a \in \tilde{A}^{\text{pa}}$) and identify \tilde{A}^{pa} with its image in \tilde{A}^P . Since $\phi(A^{\text{pa}}) \subset A^P$, we obtain the injective algebra homomorphism $\phi : A^{\text{pa}} \rightarrow A^P$ and identify A^{pa} with its image in A^P .

In this paragraph, we assume that A is an Ore domain. For each $\lambda \in P$, define the multiplicative subset S_λ of A^{pa} by $S_\lambda = \{a \in A^{\text{pa}} \mid \phi_\lambda(a) \neq 0\}$. Then S_λ is an Ore set in A^{pa} . (See the last paragraph of Remark 1.1.) Thus we obtain the subalgebra $A^{\text{pa}}[S_\lambda^{-1}]$ of K^{pa} and $\phi_\lambda : A^{\text{pa}} \rightarrow A$ is uniquely extended to the algebra homomorphism $\phi_\lambda : \tilde{A}^{\text{pa}}[S_\lambda^{-1}] \rightarrow K$. Any element of $\tilde{A}^{\text{pa}}[S_\lambda^{-1}]$ can be expressed as as^{-1} by some $a, s \in A^{\text{pa}}$ with $\phi_\lambda(s) \neq 0$ and $\phi_\lambda(as^{-1}) = \phi_\lambda(a)\phi_\lambda(s)^{-1}$. Let $A_{(P)}^{\text{pa}}$ be the intersection of $A^{\text{pa}}[S_\lambda^{-1}]$ for all $\lambda \in P$. Then $\phi_\lambda : A^{\text{pa}} \rightarrow A$ is uniquely extended to the algebra homomorphism $\phi_\lambda : A_{(P)}^{\text{pa}} \rightarrow K$. Thus we obtain the injective algebra homomorphism $\phi : A_{(P)}^{\text{pa}} \rightarrow K^P$ by $\phi(a) = (\phi_\lambda(a))_{\lambda \in P}$ ($a \in A_{(P)}^{\text{pa}}$) and identify $A_{(P)}^{\text{pa}}$ with its image in K^P .

We have $A^{\text{pa}} \subset \tilde{A}^{\text{pa}} \subset \tilde{A}^P$ in any case and $\tilde{A}^{\text{pa}} \subset A_{(P)}^{\text{pa}} \subset K^P$ in the case where A is an Ore domain.

1.4 Quantum difference operator algebras and τ -variables

In this subsection, we shall introduce difference operators acting on the parameter variables and call them τ -variables.

In the Kac-Moody case, for each $\mu \in P$, let τ^μ be the difference operator acting on quantum algebras with parameter variables given by

$$\tau^\mu(f_i) = f_i, \quad \tau^\mu(\beta) = \beta + \langle \beta, \mu \rangle \quad (i \in I, \beta \in Q^\vee).$$

Thus we obtain the difference operator algebras $D(A^{\text{pa}}) = A^{\text{pa}}[\tau^\mu | \mu \in P]$, $D(\tilde{A}^{\text{pa}}) = \tilde{A}^{\text{pa}}[\tau^\mu | \mu \in P]$, $D(K^{\text{pa}}) = K^{\text{pa}}[\tau^\mu | \mu \in P]$. In these algebras, we have

$$\tau^\lambda \tau^\mu = \tau^{\lambda+\mu}, \quad \tau^\mu f_i = f_i \tau^\mu, \quad \tau^\mu \beta = (\beta + \langle \beta, \mu \rangle) \tau^\mu \quad (\lambda, \mu \in P, \beta \in Q^\vee).$$

In the q -difference case, for each $\mu \in P$, let τ^μ be the q -difference operator acting on quantum algebras with parameter variables given by

$$\tau^\mu(f_i) = f_i, \quad \tau^\mu(q^\beta) = q^{\beta + \langle \beta, \mu \rangle} \quad (i \in I, \beta \in Q^\vee).$$

Thus we obtain the q -difference operator algebras $D(A^{\text{pa}})$, $D(\tilde{A}^{\text{pa}})$, $D(K^{\text{pa}})$, similarly as in the Kac-Moody case. In these algebras we have

$$\tau^\lambda \tau^\mu = \tau^{\lambda+\mu}, \quad \tau^\mu f_i = f_i \tau^\mu, \quad \tau^\mu q^\beta = q^{\beta + \langle \beta, \mu \rangle} \tau^\mu \quad (\lambda, \mu \in P, \beta \in Q^\vee).$$

Note that $D(K^{\text{pa}})$ is defined only in the case where A is an Ore domain.

For any algebra R and any $\mu \in P$, the algebra automorphism τ^μ of R^P is defined by the translation $\tau^\mu((a_\lambda)_{\lambda \in P}) = (a_{\lambda+\mu})_{\lambda \in P}$ ($(a_\lambda)_{\lambda \in P} \in R^P$). Thus we obtain the extended algebra $D(R^P) = R^P[\tau^\mu | \mu \in P]$.

The injective algebra homomorphism $\phi : \tilde{A}^{\text{pa}} \rightarrow \tilde{A}^P$ commutes with τ^μ for any $\mu \in P$. Therefore ϕ is naturally extended to the injective algebra homomorphism $\phi : D(\tilde{A}^{\text{pa}}) \rightarrow D(\tilde{A}^P)$. We identify $D(\tilde{A}^{\text{pa}})$ with its image in $D(\tilde{A}^P)$.

Similarly, when A is an Ore domain, $\phi : A_{(P)}^{\text{pa}} \rightarrow K$ is naturally extended to the injective algebra homomorphism $\phi : D(A_{(P)}^{\text{pa}}) \rightarrow D(K^P)$. We identify $D(\tilde{A}_{(P)}^{\text{pa}})$ with its image in $D(K^P)$.

We have $D(A^{\text{pa}}) \subset D(\tilde{A}^{\text{pa}}) \subset D(\tilde{A}^P)$ in any case and $D(\tilde{A}^{\text{pa}}) \subset D(A_{(P)}^{\text{pa}}) \subset D(K^P)$ in the case where A is an Ore domain. These algebras are called *the quantum difference operator algebras*.

We call τ^μ 's *the quantum Laurent τ -monomials*. We define *the quantum τ -variables* τ_i ($i \in I$) by $\tau_i = \tau^{\Lambda_i}$.

1.5 Quantum algebras with fractional powers

For each $\beta \in Q^\vee$, we define *the fractional power* f_i^β by $f_i^\beta = (f_i^{\langle \beta, \lambda \rangle})_{\lambda \in P} \in \tilde{A}^P$. Let \mathcal{A} be the subalgebra of \tilde{A}^P generated by $\tilde{A}^{\text{pa}} = \phi(\tilde{A}^{\text{pa}})$ and the fractional powers f_i^β ($\beta \in Q^\vee$), and $D(\mathcal{A})$ the subalgebra of $D(\tilde{A}^P)$ generated by $D(\tilde{A}^{\text{pa}}) = \phi(D(\tilde{A}^{\text{pa}}))$ and the fractional powers f_i^β ($\beta \in Q^\vee$):

$$\mathcal{A} = \tilde{A}^{\text{pa}}[f_i^\beta | \beta \in Q^\vee], \quad D(\mathcal{A}) = \mathcal{A}[\tau^\mu | \mu \in P].$$

These algebras are called *the quantum algebra with fractional powers* and *the quantum difference operator algebra with fractional powers*, respectively. In $D(\mathcal{A})$, we have

$$\tau^\mu f_i^\beta = f_i^{\beta + \langle \beta, \mu \rangle} \tau^\mu \quad (\mu \in P, \beta \in Q^\vee, i \in I).$$

For $\lambda \in P$, let $\phi_\lambda : \mathcal{A} \rightarrow \tilde{A}$ be the restriction on \mathcal{A} of the canonical projection from \tilde{A}^P onto its λ -factor. Then we have $\phi_\lambda(f_i^\beta) = f_i^{(\beta, \lambda)}$.

Similarly, when A is an Ore domain, we can construct the algebras

$$\mathcal{A}_{(P)} = A_{(P)}^{\text{pa}}[f_i^\beta | \beta \in Q^\vee], \quad D(\mathcal{A}_{(P)}) = \mathcal{A}_{(P)}[\tau^\mu | \mu \in P]$$

as subalgebras of K^P and $D(K^P)$, respectively. For $\lambda \in P$, let $\phi_\lambda : \mathcal{A}_{(P)} \rightarrow K$ be the restriction on $\mathcal{A}_{(P)}$ of the canonical projection from K^P onto its λ -factor.

1.6 Tilde action of the Weyl group

For any algebra R and any $w \in W$, the algebra automorphism \tilde{w} of $D(R^P)$ is given by

$$\tilde{w}((a_\lambda)_{\lambda \in P}) = (a_{w^{-1}(\lambda)})_{\lambda \in P}, \quad \tilde{w}(\tau^\mu) = \tau^{w(\mu)} \quad ((a_\lambda)_{\lambda \in P} \in R^P, \mu \in P).$$

This is called *the tilde action* of the Weyl group.

In the case of $R = \tilde{A}$, the tilde action of $w \in W$ on $D(\tilde{A}^P)$ preserves $D(\mathcal{A})$ and its action on $D(\mathcal{A})$ is characterized by

$$\begin{aligned} \tilde{w}(f_i^{\pm 1}) &= f_i^{\pm 1} \quad (i \in I), \\ \tilde{w}(\beta) &= w(\beta) \quad (\beta \in Q^\vee) \quad \text{in the Kac-Moody case,} \\ \tilde{w}(q^\beta) &= q^{w(\beta)} \quad (\beta \in Q^\vee) \quad \text{in the } q\text{-difference case,} \\ \tilde{w}(f_i^\beta) &= f_i^{w(\beta)} \quad (i \in I, \beta \in Q^\vee), \\ \tilde{w}(\tau^\mu) &= \tau^{w(\mu)} \quad (\mu \in P). \end{aligned}$$

Similarly, when A is an Ore domain, the tilde action on $D(K^P)$ preserves $D(\mathcal{A}_{(P)})$.

1.7 Quantum birational Weyl group action

We are ready to construct the quantization of the birational Weyl group action arising from a nilpotent Poisson algebra proposed by Noumi and Yamada [25].

Lemma 1.5 (Verma identities). *In $D(\mathcal{A})$, for any $\beta, \gamma \in Q^\vee$, we have*

- If $(a_{ij}, a_{ji}) = (0, 0)$, then $f_i^\beta f_j^\gamma = f_j^\gamma f_i^\beta$.
- If $(a_{ij}, a_{ji}) = (-1, -1)$, then $f_i^\beta f_j^{\beta+\gamma} f_i^\gamma = f_j^\gamma f_i^{\beta+\gamma} f_j^\beta$.
- If $(a_{ij}, a_{ji}) = (-1, -2)$, then $f_i^\beta f_j^{2\beta+\gamma} f_i^{\beta+\gamma} f_j^\gamma = f_j^\gamma f_i^{\beta+\gamma} f_j^{2\beta+\gamma} f_i^\beta$.
- If $(a_{ij}, a_{ji}) = (-1, -3)$, then $f_i^\beta f_j^{3\beta+\gamma} f_i^{2\beta+\gamma} f_j^{3\beta+2\gamma} f_i^{\beta+\gamma} f_j^\gamma = f_j^\gamma f_i^{\beta+\gamma} f_j^{3\beta+2\gamma} f_i^{2\beta+\gamma} f_j^{3\beta+\gamma} f_i^\beta$.

Proof. It is sufficient to show that the identities obtained by substituting (m, n) to (β, γ) in the above identities hold for all integers m, n . But it reduces to the cases where m, n are non-negative. The proof of the non-negative cases is found in Proposition 39.3.7 of [17]. \square

Remark 1.6. The Verma identities (Proposition 39.3.7 of [17]) can be regarded as a corollary of the uniqueness (up to scalar multiples) of homomorphisms between Verma modules. See 4.4.16 of [13]. \square

Example 1.7. Under the setting of Example 1.3 and $Q^\vee = \mathbb{Z}\alpha_1^\vee \oplus \mathbb{Z}\alpha_2^\vee$, the following formula holds in $\mathcal{A}^{\text{pa}} = \mathbb{C}[x, \partial, \alpha_i^\vee, x^{\pm\alpha_i^\vee}, \partial^{\pm\alpha_i^\vee} | i = 1, 2]$:

$$x^a \partial^{a+b} x^b = \partial^b x^{a+b} \partial^a, \quad (1.1)$$

where $a = \beta$, $b = \gamma$, and $\beta, \gamma \in Q^\vee$. Although this is a special case of Lemma 1.5, we shall show the direct proof to help understanding the proof of Lemma 1.5. It is sufficient to prove that the formula (1.1) holds for any $a, b \in \mathbb{Z}$. Using the Leibnitz formula

$$\partial^n f = \sum_{k=0}^n \binom{n}{k} f^{(k)} \partial^{n-k} \quad \text{in } \mathbb{C}[x, \partial] \quad (f \in \mathbb{C}[x], n \in \mathbb{Z}_{\geq 0}),$$

we can show that for any $a, b \in \mathbb{Z}_{\geq 0}$, the both-hand sides of (1.1) are equal to

$$\sum_{k=0}^b k! \binom{a+b}{k} \binom{b}{k} x^{a+b-k} \partial^{a+b-k}.$$

The formula (1.1) for $a, b \in \mathbb{Z}_{\geq 0}$ has been proved. Replacing (a, b) with $(a+b, -b)$, we find that the formula (1.1) for $a, b \in \mathbb{Z}_{\geq 0}$ is equivalent to the one for $a \in \mathbb{Z}_{\geq 0}$ and $b \in \mathbb{Z}_{\leq 0}$ with $a+b \geq 0$. Replacing (a, b) with $(-a-b, a)$, we find that the formula (1.1) for $a, b \in \mathbb{Z}_{\geq 0}$ is equivalent to the one for $a \in \mathbb{Z}_{\geq 0}$ and $b \in \mathbb{Z}_{\leq 0}$ with $a+b \leq 0$. We can similarly obtain the other cases and prove the formula (1.1) for any $a, b \in \mathbb{Z}$.

The fractional power ∂^β of $\partial = d/dx$ is related to the fractional calculus and the Kats middle convolution. See [28], [21], and references therein. \square

Remark 1.8. For $c = q^\beta$ ($\beta \in Q^\vee$), define \mathbf{e}_i^c by $\mathbf{e}_i^c(a) = f_i^\beta a f_i^{-\beta}$. Then Lemma 1.5 immediately leads to the Verma identities of \mathbf{e}_i^c ($i \in I$): for $c = q^\beta$ and $d = q^\gamma$,

- If $(a_{ij}, a_{ji}) = (0, 0)$, then $\mathbf{e}_i^c \mathbf{e}_j^d = \mathbf{e}_j^d \mathbf{e}_i^c$.
- If $(a_{ij}, a_{ji}) = (-1, -1)$, then $\mathbf{e}_i^c \mathbf{e}_j^{cd} \mathbf{e}_i^d = \mathbf{e}_j^c \mathbf{e}_i^{cd} \mathbf{e}_j^c$.
- If $(a_{ij}, a_{ji}) = (-1, -2)$, then $\mathbf{e}_i^c \mathbf{e}_j^{c^2 d} \mathbf{e}_i^{cd} \mathbf{e}_j^d = \mathbf{e}_j^d \mathbf{e}_i^{cd} \mathbf{e}_j^{c^2 d} \mathbf{e}_i^c$.
- If $(a_{ij}, a_{ji}) = (-1, -3)$, then $\mathbf{e}_i^c \mathbf{e}_j^{c^3 d} \mathbf{e}_i^{c^2 d^2} \mathbf{e}_j^{cd^2} \mathbf{e}_i^{cd} \mathbf{e}_j^d = \mathbf{e}_j^c \mathbf{e}_i^{cd} \mathbf{e}_j^{c^3 d^2} \mathbf{e}_i^{c^2 d} \mathbf{e}_j^{c^2 d} \mathbf{e}_i^c$.

These identities mean that $\{\mathbf{e}_i^c\}_{i \in I}$ can be regarded as a quantum version of geometric crystal defined by Berenstein and Kazhdan [1]. \square

For any associative algebra R and an invertible element $a \in R^\times$, the inner algebra automorphism $\text{Ad}(a)$ of R is defined by

$$\text{Ad}(a)(x) = axa^{-1} \quad \text{for } x \in R.$$

Then the multiplicative group R^\times acts on R via Ad .

Definition 1.9. For each $i \in I$, the algebra automorphisms \mathbf{s}_i of $D(\mathcal{A})$ (and of $D(\mathcal{A}_{(P)})$ when A is an Ore domain) by $\mathbf{s}_i = \text{Ad}(f_i^{\alpha_i^\vee})\tilde{s}_i$. \square

Lemma 1.10. The set $\{\mathbf{s}_i\}_{i \in I}$ of algebra automorphisms of $D(\mathcal{A})$ (and of $D(\mathcal{A}_{(P)})$ when A is an Ore domain) satisfies the fundamental relations of the Weyl group: $\mathbf{s}_i\mathbf{s}_j = \mathbf{s}_j\mathbf{s}_i$ if $(a_{ij}, a_{ji}) = (0, 0)$; $\mathbf{s}_i\mathbf{s}_j\mathbf{s}_i = \mathbf{s}_j\mathbf{s}_i\mathbf{s}_j$ if $(a_{ij}, a_{ji}) = (-1, -1)$; $(\mathbf{s}_i\mathbf{s}_j)^2 = (\mathbf{s}_j\mathbf{s}_i)^2$ if $(a_{ij}, a_{ji}) = (-1, -2)$; $(\mathbf{s}_i\mathbf{s}_j)^3 = (\mathbf{s}_j\mathbf{s}_i)^3$ if $(a_{ij}, a_{ji}) = (-1, -3)$; $\mathbf{s}_i^2 = 1$.

Proof. The relation $\mathbf{s}_i^2 = 1$ follows from $\tilde{s}_i(f_i^{\alpha_i^\vee}) = f_i^{-\alpha_i^\vee}$:

$$\mathbf{s}_i^2 = \text{Ad}(f_i^{\alpha_i^\vee})\tilde{s}_i \text{Ad}(f_i^{\alpha_i^\vee})\tilde{s}_i = \text{Ad}(f_i^{\alpha_i^\vee}) \text{Ad}(f_i^{-\alpha_i^\vee})\tilde{s}_i\tilde{s}_i = \text{Ad}(f_i^{\alpha_i^\vee} f_i^{-\alpha_i^\vee})\tilde{s}_i^2 = 1.$$

The other fundamental relations are no more than rewrites of the Verma identities (Lemma 1.5). For example, in the case of $(a_{ij}, a_{ji}) = (-1, -1)$, the relation $\mathbf{s}_i\mathbf{s}_j\mathbf{s}_i = \mathbf{s}_j\mathbf{s}_i\mathbf{s}_j$ is proved as below. Using the formula $\tilde{w}(f_k^\beta) = f_k^{w(\beta)}$ ($w \in W$, $k \in I$, $\beta \in Q^\vee$), we obtain

$$\mathbf{s}_i\mathbf{s}_j\mathbf{s}_i = \text{Ad}(f_i^{\alpha_i^\vee} f_j^{\alpha_j^\vee + \alpha_j^\vee} f_i^{\alpha_i^\vee})\tilde{s}_i\tilde{s}_j\tilde{s}_i, \quad \mathbf{s}_j\mathbf{s}_i\mathbf{s}_j = \text{Ad}(f_j^{\alpha_j^\vee} f_i^{\alpha_i^\vee + \alpha_i^\vee} f_j^{\alpha_j^\vee})\tilde{s}_j\tilde{s}_i\tilde{s}_j.$$

Therefore the relation $\mathbf{s}_i\mathbf{s}_j\mathbf{s}_i = \mathbf{s}_j\mathbf{s}_i\mathbf{s}_j$ follows from the Verma identity for $(a_{ij}, a_{ji}) = (-1, -1)$. The other relations are proved by the same argument. \square

This lemma immediately leads to the following theorem.

Theorem 1.11. The mapping $s_i \mapsto \mathbf{s}_i$ ($i \in I$) induces the Weyl group actions on $D(\mathcal{A})$ (and on $D(\mathcal{A}_{(P)})$ when A is an Ore domain). \square

This theorem can be regarded as a both q -difference and canonically quantized version of Theorem 1.1 and Theorem 1.2 of [25].

Definition 1.12. We call the Weyl group actions obtained in Theorem 1.11 *the quantum birational Weyl group actions* and denote by $w(x)$ the quantum birational action of $w \in W$ on x . \square

Remark 1.13. In [8], using the quantum dilogarithm, Hasegawa quantizes certain birational Weyl group actions of q -difference type proposed by Kajiwara, Noumi, and Yamada [15]. Although the quantum birational Weyl group actions of Hasegawa are different from ours, they can be also reconstructed by the method of fractional powers (Section 5 of [16]). The quantum τ -functions for the Hasegawa actions are not discovered at the present time. \square

1.8 Explicit formulas for the action

The following formulas immediately follow from the definition of \mathbf{s}_i and the quantum birational Weyl group action (Definition 1.12):

$$s_i(\beta) = \beta - \langle \beta, \alpha_i \rangle \alpha_i^\vee \quad (\beta \in Q^\vee) \quad \text{in the Kac-Moody case,} \quad (1.2)$$

$$s_i(q^\beta) = q^{s_i(\beta)} = q^{\beta - \langle \beta, \alpha_i \rangle \alpha_i^\vee} \quad (\beta \in Q^\vee) \quad \text{in the } q\text{-difference case,} \quad (1.3)$$

$$s_i(\tau^\mu) = f_i^{\langle \alpha_i^\vee, \mu \rangle} \tau^{s_i(\mu)} = f_i^{\langle \alpha_i^\vee, \mu \rangle} \tau^{\mu - \langle \alpha_i^\vee, \mu \rangle \alpha_i} \quad (\mu \in P), \quad (1.4)$$

$$s_i(f_i) = f_i. \quad (1.5)$$

The second last formula follows from $\tau^\mu f_i^{\alpha_i^\vee} = f_i^{\alpha_i^\vee + \langle \alpha_i^\vee, \mu \rangle} \tau^\mu$ and $\langle \alpha_i^\vee, s_i(\mu) \rangle = -\langle \alpha_i^\vee, \mu \rangle$:

$$s_i(\tau^\mu) = f_i^{\alpha_i^\vee} \tau^{s_i(\mu)} f_i^{-\alpha_i^\vee} = f_i^{\alpha_i^\vee} f_i^{-\alpha_i^\vee + \langle \alpha_i^\vee, \mu \rangle} \tau^{s_i(\mu)} = f_i^{\langle \alpha_i^\vee, \mu \rangle} \tau^{s_i(\mu)}.$$

In particular, we have $s_i(\tau_i) = f_i \tau^{\Lambda_i - \alpha_i}$ and $s_i(\tau_j) = \tau_j$ ($i \neq j$).

Remark 1.14. Since $f_j = s_j(\tau_j) \tau^{-\Lambda_j + \alpha_j}$, we have

$$s_i(f_j) = s_i s_j(\tau_j) s_i(\tau^{-\Lambda_j + \alpha_j}) \quad (i, j \in I).$$

This means that the quantum birational Weyl group action on the dependent variables f_i ($i \in I$) is described by the action on the quantum Laurent τ -monomials τ^μ ($\mu \in P$). This observation is one of the motivation of introducing the quantum τ -functions. For the definition of them, see Section 2.1. \square

We shall write down the explicit formulas of $s_i(f_j)$ for $i \neq j$ as follows.

In the Kac-Moody case, we define the commutator $[A, B]$ by $[A, B] = AB - BA$ and $\text{ad } f : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ for $f \in \mathfrak{g}$ by

$$(\text{ad } f)(a) = fa - af \quad (a \in U(\mathfrak{g})).$$

Then the Serre relations are rewritten in the form $(\text{ad } f_i)^{1-a_{ij}}(f_j) = 0$ ($i \neq j$). More generally, we have

$$(\text{ad } f)^k(a) = \sum_{s=0}^k (-1)^s \binom{k}{s} f^{k-s} a f^s \quad (k \in \mathbb{Z}_{\geq 0}).$$

By induction on $|n|$, we can obtain

$$f_i^n f_j f_i^{-n} = \sum_{k=0}^{-a_{ij}} \binom{n}{k} (\text{ad } f_i)^k(f_j) f_i^{-k} \quad (i \neq j, n \in \mathbb{Z}).$$

It immediately follows that

$$f_i^\beta f_j f_i^{-\beta} = \sum_{k=0}^{-a_{ij}} \binom{\beta}{k} (\text{ad } f_i)^k(f_j) f_i^{-k} \in \tilde{A}^{\text{pa}} \quad (i \neq j, \beta \in Q^\vee).$$

In particular, we have

$$s_i(f_j) = f_i^{\alpha_i^\vee} f_j f_i^{-\alpha_i^\vee} = \sum_{k=0}^{-a_{ij}} \binom{\alpha_i^\vee}{k} (\text{ad } f_i)^k(f_j) f_i^{-k} \in \tilde{A}^{\text{pa}} \quad (i \neq j). \quad (1.6)$$

This result is a canonically quantized version of the formula (1.9) of [25] specialized by $\psi = \varphi_j$.

In the q -difference case, we define the q -commutator $[A, B]_q$ by $[A, B]_q = AB - qBA$ and $\text{ad } f_i : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ by

$$(\text{ad } f_i)(a) = f_i a - q_i^{-h_i} a q_i^{h_i} f_i \quad (a \in U_q(\mathfrak{g})).$$

Then the q -Serre relations are also rewritten in the form $(\text{ad } f_i)^{1-a_{ij}}(f_j) = 0$ ($i \neq j$). More generally, we have

$$\begin{aligned} (\text{ad } f_i)^k(f_j) &= [f_i, [\cdots, [f_i, [f_i, f_j]_{q_i^{a_{ij}}} \cdots]_{q_i^{a_{ij}+2}} \cdots]_{q_i^{a_{ij}+2(k-2)}}]_{q_i^{a_{ij}+2(k-1)}} \\ &= \sum_{s=0}^k (-1)^s q_i^{s(s+a_{ij}-1)} \begin{bmatrix} k \\ s \end{bmatrix}_{q_i} f_i^{k-s} f_j f_i^s \quad (k \in \mathbb{Z}_{\geq 0}). \end{aligned}$$

By induction on $|n|$, we can obtain

$$f_i^n f_j f_i^{-n} = \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} (\text{ad } f_i)^k(f_j) f_i^{-k} \quad (i \neq j, n \in \mathbb{Z}).$$

For the derivation of this formula, see also Chapter 7 of [17]. It immediately follows that

$$s_i(f_j) = f_i^\beta f_j f_i^{-\beta} = \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\beta-k)} \begin{bmatrix} \beta \\ k \end{bmatrix}_{q_i} (\text{ad } f_i)^k(f_j) f_i^{-k} \in \widetilde{A}^{\text{pa}} \quad (i \neq j, \beta \in Q^\vee).$$

In particular, we have

$$s_i(f_j) = f_i^{\alpha_i^\vee} f_j f_i^{-\alpha_i^\vee} = \sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\alpha_i^\vee-k)} \begin{bmatrix} \alpha_i^\vee \\ k \end{bmatrix}_{q_i} (\text{ad } f_i)^k(f_j) f_i^{-k} \in \widetilde{A}^{\text{pa}} \quad (i \neq j). \quad (1.7)$$

This result is a both q -difference and canonically quantized version of the formula (1.9) of [25] specialized by $\psi = \varphi_j$.

Thus we obtain the following lemma.

Lemma 1.15. *For any $\beta \in Q^\vee$ and $i, j \in I$, we have $f_i^\beta f_j f_i^{-\beta} \in \widetilde{A}^{\text{pa}}$. More precisely, $f_i^\beta f_j f_i^{-\beta}$ belongs to the subalgebra of $\widetilde{A}^{\text{pa}}$ generated by $\{f_i^{\pm 1}, f_j, \beta\}$ in the Kac-Moody case and by $\{f_i^{\pm 1}, f_j, q_i^{\pm \beta}\}$ in the q -difference case. In particular, we have $s_i(f_j) \in \widetilde{A}^{\text{pa}}$. \square*

Example 1.16. Suppose that $a_{ij} = -1$ and $d_i = 1$. In the Kac-Moody case, we have

$$s_i(f_j) = f_j + \alpha_i^\vee [f_i, f_j] f_i^{-1} = (1 - \alpha_i^\vee) f_j + \alpha_i^\vee f_i f_j f_i^{-1}.$$

In the q -difference case, we have

$$s_i(f_j) = q^{-\alpha_i^\vee} f_j + [\alpha_i^\vee]_q [f_i, f_j]_{q^{-1}} f_i^{-1} = [1 - \alpha_i^\vee]_q f_j + [\alpha_i^\vee]_q f_i f_j f_i^{-1}.$$

This formula shall be used in Example 2.1. \square

Remark 1.17. Suppose that A is an Ore domain. It follows from Lemma 1.15 that the quantum birational Weyl group action preserves $D(A_{(P)}^{\text{pa}})$ and $A_{(P)}^{\text{pa}}$. Therefore W acts on them. This result is an extended version of Theorem 4.3 of [16]. Since the algebra $D(A_{(P)}^{\text{pa}})$ does not contain the fractional powers f_i^β , the quantum birational Weyl group action on $D(A_{(P)}^{\text{pa}})$ is characterized by the explicit formulas written down in this subsection. \square

Example 1.18. Under the setting of Example 1.7, we have

$$\begin{aligned} s_1(x) &= x, & s_1(\partial) &= x^{\alpha_1^\vee} \partial x^{-\alpha_1^\vee} = \partial - \frac{\alpha_1^\vee}{x}, \\ s_2(x) &= \partial^{\alpha_2^\vee} x \partial^{-\alpha_2^\vee} = x + \alpha_2^\vee \partial^{-1}, & s_2(\partial) &= \partial, \\ s_i(\alpha_i^\vee) &= -\alpha_i^\vee, & s_1(\alpha_2^\vee) &= s_2(\alpha_1^\vee) = \alpha_1^\vee + \alpha_2^\vee. \end{aligned}$$

These formulas define the action of $W = S_3 = \langle s_1, s_2 \rangle$ on $A_{(P)}^{\text{pa}} = \mathbb{C}[x, \partial, \alpha_1^\vee, \alpha_2^\vee]_{(P)}$. A non-commutative rational function $a \in \mathbb{C}(x, \partial, \alpha_1^\vee, \alpha_2^\vee)$ is an element of $\mathbb{C}[x, \partial, \alpha_1^\vee, \alpha_2^\vee]_{(P)}$ if and only if, for any $\lambda \in P$, there exist some $b, s \in \mathbb{C}[x, \partial, \alpha_1^\vee, \alpha_2^\vee]$ such that $a = bs^{-1}$ and s does not vanish even if $\langle \alpha_i^\vee, \lambda \rangle$'s are substituted into α_i^\vee 's. \square

2 Quantum τ -functions

2.1 Definition of quantum τ -functions

The Tits cone is defined to be $WP_+ = \{w(\mu) \mid w \in W, \mu \in P_+\} \subset P$. For each $\nu \in WP_+$, we define the quantum τ -function $\tau_{(\nu)}$ by

$$\tau_{(\nu)} = w(\tau^\mu), \quad \nu = w(\mu), \quad w \in W, \quad \mu \in P_+.$$

Note that $w(\tau^\mu)$ depends only on $\nu = w(\mu)$ owing to the property $s_i(\tau_j) = \tau_j$ ($i \neq j$) of the quantum birational Weyl group action.

Example 2.1. In the q -difference case, if $i \neq j$, then we have

$$\begin{aligned} \tau_{(\Lambda_j)} &= \tau_j, & \tau_{(s_j(\Lambda_j))} &= s_j(\tau_j) = f_j \tau^{s_j(\Lambda_j)}, \\ \tau_{(s_i s_j(\Lambda_j))} &= s_i s_j(\tau_j) = f_i^{\alpha_i^\vee} f_j \tau^{s_i s_j(\Lambda_j)} f_i^{-\alpha_i^\vee} = f_i^{\alpha_i^\vee} f_j f_i^{-\alpha_i^\vee - a_{ij}} \tau^{s_i s_j(\Lambda_i)} \\ &= \left(\sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\alpha_i^\vee - k)} \begin{bmatrix} \alpha_i^\vee \\ k \end{bmatrix}_{q_i} (\text{ad } f_i)^k (f_j) f_i^{-a_{ij}-k} \right) \tau^{s_i s_j(\Lambda_i)}. \end{aligned}$$

Note that the quantum τ -function $\tau_{(s_i s_j(\Lambda_j))}$ is a polynomial in f_i, f_j and a Laurent polynomial in $q_i^{\alpha_i^\vee}$. In particular, when $a_{ij} = -1$ and $d_i = 1$, we have

$$\begin{aligned} \tau_{(s_i s_j(\Lambda_j))} &= s_i s_j(\tau_j) = \left(q^{-\alpha_i^\vee} f_j f_i + [\alpha_j^\vee]_q [f_i, f_j]_{q^{-1}} \right) \tau^{s_j s_i(\Lambda_i)} \\ &= ([1 - \alpha_i^\vee]_q f_j f_i + [\alpha_i^\vee]_q f_i f_j) \tau^{s_j s_i(\Lambda_i)}. \end{aligned}$$

The last expression shall be used for the proof of the quantum q -Hirota-Miwa equation in Section 2.2. \square

Example 2.2. Assume that the GCM is of type A_3 : $I = \{1, 2, 3\}$, $a_{ii} = 2$, $a_{i,i+1} = a_{i+1,i} = -1$ ($i = 1, 2$), $a_{ij} = 0$ ($|i - j| \geq 2$), $d_i = 1$. Let A be the associative algebra over \mathbb{C} generated by f_1, f_2, f_3 with fundamental relations $[f_1, f_2] = [f_2, f_3] = 1$, $[f_1, f_3] = 0$.

We set $(i_1, i_2, \dots, i_6) = (1, 2, 3, 1, 2, 1)$, $w_k = s_{i_k} \cdots s_{i_2} s_{i_1} \in W$, and $\beta_k := w_{k-1}^{-1}(\alpha_{i_k}^\vee) = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}^\vee)$ ($k = 1, 2, \dots, 6$). We have $\beta_1 = \alpha_1^\vee$, $\beta_2 = \alpha_1^\vee + \alpha_2^\vee$, $\beta_3 = \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee$, $\beta_4 = \alpha_2^\vee$, $\beta_5 = \alpha_2^\vee + \alpha_3^\vee$, and $\beta_6 = \alpha_3^\vee$.

Then the quantum τ -functions $\tau_{(w_k(\Lambda_1))}$ ($k = 1, 2, \dots, 6$) can be written in the form $\tau_{(w_k(\Lambda_1))} = \tilde{w}_k(X_k) \tau^{w_k(\Lambda_1)}$, where X_k ($k = 1, 2, \dots, 6$) are calculated as follows:

$$\begin{aligned} X_1 &= f_1^{-\beta_1} f_1^{\beta_1+1} = f_1, \\ X_2 &= f_2^{-\beta_2} X_1 f_2^{\beta_2+1} = \left(f_1 + \frac{\beta_2}{f_2}\right) f_2 = f_1 f_2 + \beta_2, \\ X_3 &= f_3^{-\beta_3} X_2 f_3^{\beta_3+1} = \left(f_1 \left(f_2 + \frac{\beta_3}{f_3}\right) + \beta_2\right) f_3 = f_1 f_2 f_3 + \beta_3 f_1 + \beta_2 f_3, \\ X_4 &= f_1^{-\beta_4} X_3 f_1^{\beta_4} = f_1 \left(f_2 - \frac{\beta_4}{f_1}\right) f_3 + \beta_3 f_1 + \beta_2 f_3 = f_1 f_2 f_3 + \beta_3 f_1 + (\beta_2 - \beta_4) f_3, \\ X_5 &= f_2^{-\beta_5} X_4 f_2^{\beta_5} = \left(f_1 + \frac{\beta_5}{f_2}\right) f_2 \left(f_3 - \frac{\beta_5}{f_2}\right) + \beta_3 \left(f_1 + \frac{\beta_5}{f_2}\right) + (\beta_2 - \beta_4) \left(f_3 - \frac{\beta_5}{f_2}\right) \\ &= f_1 f_2 f_3 + (\beta_3 - \beta_5) f_1 + (\beta_2 - \beta_4 + \beta_5) f_3 + \underbrace{\left(-\beta_2 + \beta_4 - \beta_5 + \beta_3\right)}_{\text{cancels out}} \frac{\beta_5}{f_2}, \\ &= f_1 f_2 f_3 + (\beta_3 - \beta_5) f_1 + (\beta_2 - \beta_4 + \beta_5) f_3, \\ X_6 &= f_1 f_2 f_3 + \beta_6 f_1 + (\beta_3 - \beta_6) f_3. \end{aligned}$$

Thus the all quantum τ -functions $\tau_{(w_k(\Lambda_1))}$ ($k = 1, 2, \dots, 6$) are polynomials in f_i and α_i^\vee ($i \in I$). \square

2.2 The quantum q -Hirota-Miwa equation

In this subsection, as a supporting evidence for the correctness of the definition of the quantum τ -functions in the previous subsection, we shall show that the quantum τ -functions of type $A_{n-1}^{(1)}$ satisfy the quantum q -Hirota-Miwa equation (2.1) in the q -difference case.

Assume that $n \geq 3$ and the GCM $[a_{ij}]_{i,j \in I}$ is of type $A_{n-1}^{(1)}$: $I = \mathbb{Z}/n\mathbb{Z}$, $a_{ii} = 2$, $a_{i,i\pm 1} = -1$, $a_{ij} = 0$ ($j \neq i, i \pm 1$), and $d_i = 1$. Denote the image of $k \in \mathbb{Z}$ in $I = \mathbb{Z}/n\mathbb{Z}$ by \bar{k} . Assume that the algebra automorphism of A can be defined by $f_i \mapsto f_{i+1}$ for $i \in I$.

We define the coroot lattice Q^\vee to be the free \mathbb{Z} -module generated by δ^\vee and ε_k^\vee ($k = 1, 2, \dots, n$). We set ε_k^\vee ($k \in \mathbb{Z}$) by the quasi-periodicity $\varepsilon_{k+n}^\vee = \varepsilon_k^\vee - \delta^\vee$. Define the simple coroots by $\alpha_k^\vee = \varepsilon_k^\vee - \varepsilon_{k+1}^\vee$ ($k \in \mathbb{Z}$). Then we have $\alpha_{k+n}^\vee = \alpha_k^\vee$ and put $\alpha_{\bar{k}}^\vee = \alpha_k^\vee$. Since $\alpha_0^\vee = \delta^\vee + \varepsilon_n^\vee - \varepsilon_1^\vee$, the set $\{\alpha_k^\vee\}_{k=0}^{n-1}$ is linearly independent over \mathbb{Z} and $\sum_{k=0}^{n-1} \alpha_k^\vee = \delta^\vee$.

The weight lattice P is given by $P = \text{Hom}(Q^\vee, \mathbb{Z})$. Denote by Λ_0, ε_k ($k = 1, 2, \dots, n$) the dual basis of $\delta^\vee, \varepsilon_k^\vee$ ($k = 1, 2, \dots, n$). We set ε_k ($k \in \mathbb{Z}$) by the periodicity $\varepsilon_{k+n} = \varepsilon_k$. We define ϖ_k ($k \in \mathbb{Z}$) by $\varpi_k = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k$ ($k \in \mathbb{Z}_{\geq 0}$) and the quasi-periodicity $\varpi_{k+n} = \varpi_k + \varpi_n$. Then Λ_0 and ϖ_k ($k = 1, 2, \dots, n$) span the weight lattice P . We define Λ_k ($k \in \mathbb{Z}$) by $\Lambda_k = \Lambda_0 + \varpi_k$. Then $\Lambda_{k+n} = \Lambda_k + \varpi_n$. Since $\langle \alpha_k^\vee, \varpi_n \rangle = 0$ ($k \in \mathbb{Z}$), we have $\langle \alpha_k^\vee, \Lambda_l \rangle = \delta_{\bar{k}, \bar{l}}$.

We define the simple roots α_k ($k \in \mathbb{Z}$) by $\alpha_k = -\Lambda_{k-1} + 2\Lambda_k - \Lambda_{k+1} = \varepsilon_k - \varepsilon_{k+1}$. Then we have $\alpha_{k+n} = \alpha_k$ and $\langle \alpha_k^\vee, \alpha_l \rangle = a_{\bar{k}, \bar{l}}$. Put $\alpha_{\bar{k}} = \alpha_k$. In this setting, we have $\sum_{k=0}^{n-1} \alpha_k = 0$.

Set $s_k = s_{\bar{k}}$ and $f_k = f_{\bar{k}}$ for $k \in \mathbb{Z}$. The Weyl group actions on Q^\vee and P are characterized by the following formulas:

$$\begin{aligned} s_k(\delta^\vee) &= \delta^\vee, & s_k(\varepsilon_k^\vee) &= \varepsilon_{k+1}^\vee, & s_k(\varepsilon_{k+1}^\vee) &= \varepsilon_k^\vee, & s_k(\varepsilon_l^\vee) &= \varepsilon_l^\vee \quad (\bar{l} \neq \bar{k}, \bar{k}+1), \\ s_0(\Lambda_0) &= \Lambda_{-1} - \Lambda_0 + \Lambda_1 = \Lambda_0 - \varepsilon_n + \varepsilon_1, & s_k(\Lambda_0) &= \Lambda_0 \quad (\bar{k} \neq \bar{0}), \\ s_k(\varepsilon_k) &= \varepsilon_{k+1}, & s_k(\varepsilon_{k+1}) &= \varepsilon_k, & s_k(\varepsilon_l) &= \varepsilon_l \quad (\bar{l} \neq \bar{k}, \bar{k}+1). \end{aligned}$$

We define the τ -variables τ_k ($k \in \mathbb{Z}$) by $\tau_k = \tau^{\Lambda_k}$. Then $\tau_{k+n} = \tau_k \tau^{\varpi_n}$. Note that τ^{ϖ_n} commutes with all $q^{\alpha_k^\vee}$.

Lemma 2.3. *For any $k \in \mathbb{Z}$, we have*

$$[\alpha_{k+1}^\vee]_q \tau_k s_k s_{k+1}(\tau_{k+1}) + [\alpha_k^\vee]_q s_{k+1} s_k(\tau_k) \tau_{k+1} = [\alpha_k^\vee + \alpha_{k+1}^\vee]_q s_k(\tau_k) s_{k+1}(\tau_{k+1}).$$

Proof. The definition of the quantum birational Weyl group action immediately leads to the following formulas:

$$\begin{aligned} s_k(\tau_k) &= f_k \frac{\tau_{k-1} \tau_{k+1}}{\tau_k}, & s_{k+1}(\tau_{k+1}) &= f_{k+1} \frac{\tau_k \tau_{k+2}}{\tau_{k+1}}, \\ s_k s_{k+1}(\tau_{k+1}) &= ([1 - \alpha_k^\vee]_q f_{k+1} f_k + [\alpha_k^\vee]_q f_k f_{k+1}) \frac{\tau_{k-1} \tau_{k+2}}{\tau_k}, \\ s_{k+1} s_k(\tau_k) &= ([1 - \alpha_{k+1}^\vee]_q f_k f_{k+1} + [\alpha_{k+1}^\vee]_q f_{k+1} f_k) \frac{\tau_{k-1} \tau_{k+2}}{\tau_{k+1}}. \end{aligned}$$

See Example 2.1. Using $\tau_k q^{\alpha_k^\vee} = q^{\alpha_{k+1}^\vee + 1} \tau_k$, we obtain

$$\begin{aligned} [\alpha_{k+1}^\vee]_q \tau_k s_k s_{k+1}(\tau_{k+1}) &= [\alpha_{k+1}^\vee]_q ([-\alpha_k^\vee]_q f_{k+1} f_k + [\alpha_k^\vee + 1]_q f_k f_{k+1}) \tau_{k-1} \tau_{k+2}, \\ [\alpha_k^\vee]_q s_{k+1} s_k(\tau_k) \tau_{k+1} &= [\alpha_k^\vee]_q ([1 - \alpha_{k+1}^\vee]_q f_k f_{k+1} + [\alpha_{k+1}^\vee]_q f_{k+1} f_k) \tau_{k-1} \tau_{k+2}. \end{aligned}$$

Add the right-hand sides of the two formulas. Then the $f_{k+1} f_k$ -terms cancel out and we get

$$[\alpha_k^\vee + \alpha_{k+1}^\vee]_q f_k f_{k+1} \tau_{k-1} \tau_{k+2} = [\alpha_k^\vee + \alpha_{k+1}^\vee]_q s_k(\tau_k) s_{k+1}(\tau_{k+1}).$$

The lemma has been proved. \square

Remark 2.4. In the quantum case, we must be careful to non-commutativity. In the above lemma, we have the following commutativity and non-commutativity:

1. Although τ_k and $s_k s_{k+1}(\tau_{k+1})$ does not commute, each of them commutes with $[\alpha_{k+1}^\vee]_q$.
2. Although $s_{k+1} s_k(\tau_k)$ and τ_{k+1} does not commute, each of them commutes with $[\alpha_k^\vee]_q$.
3. $s_k(\tau_k)$ and $s_{k+1}(\tau_{k+1})$ commutes. Although each of them does not commutes with $[\alpha_k^\vee + \alpha_{k+1}^\vee]_q$, their product $s_k(\tau_k) s_{k+1}(\tau_{k+1})$ commutes with $[\alpha_k^\vee + \alpha_{k+1}^\vee]_q$. \square

The extended Weyl group \widetilde{W} is defined by the semi-direct product $\widetilde{W} = W \rtimes \langle \pi \rangle$ with defining relations $\pi s_k = s_{k+1} \pi$ ($k \in \mathbb{Z}$). The actions of π on Q^\vee and P are given by $\pi(\delta^\vee) = \delta^\vee$, $\pi(\varepsilon_k^\vee) = \varepsilon_{k+1}^\vee$, $\pi(\Lambda_0) = \Lambda_1 = \Lambda_0 + \varepsilon_1$, and $\pi(\varepsilon_k) = \varepsilon_{k+1}$. These formulas define the extended Weyl group action on Q^\vee and P which preserves the canonical pairing between them. Then we have $\pi(\alpha_k^\vee) = \alpha_{k+1}^\vee$, $\pi(\varpi_k) = \varpi_{k+1} - \varepsilon_1$, and $\pi(\Lambda_k) = \Lambda_{k+1}$.

We define the actions of π on f_k and τ_k by $\pi(f_k) = f_{k+1}$ and $\pi(\tau_k) = \tau_{k+1}$. Then we have $\pi(\tau^\mu) = \tau^{\pi(\mu)}$ ($\mu \in P$).

We define $T_k \in \widetilde{W}$ ($i \in \mathbb{Z}$) by $T_k = s_{k-1} \cdots s_2 s_1 \pi s_{n-1} s_{n-2} \cdots s_k$ ($k = 1, 2, \dots, n$) and the periodicity $T_{k+n} = T_k$. Then T_k ($k \in \mathbb{Z}$) mutually commute and we have

$$\begin{aligned} s_k T_k s_k^{-1} &= T_{k+1}, & s_k T_{k+1} s_k^{-1} &= T_k, & s_k T_l s_k^{-1} &= T_l \quad (\bar{l} \neq \bar{k}, \bar{k} + 1), & \pi T_k \pi^{-1} &= T_{k+1}, \\ T_k(\delta^\vee) &= \delta^\vee, & T_k(\varepsilon_l^\vee) &= \varepsilon_l^\vee - \delta_{\bar{k}, \bar{l}} \delta^\vee, & T_k(\varepsilon_l) &= \varepsilon_l, & T_k(\Lambda_0) &= \Lambda_0 + \varepsilon_k. \end{aligned}$$

For $m = \sum_{k=1}^n m_k \varepsilon_k \in L = \bigoplus_{k=1}^n \mathbb{Z} \varepsilon_k$, we put $T^m = \prod_{k=1}^n T_k^{m_k}$. Then we have $T^m(\Lambda_k) = \Lambda_k + m$.

We define $\varepsilon_k^\vee(m)$, $\alpha_k^\vee(m)$, and $\tau_k(m)$ for $m \in L$ by

$$\begin{aligned} \varepsilon_k^\vee(m) &= T^m(\varepsilon_k^\vee) = \varepsilon_k^\vee - m_k \delta^\vee, & \alpha_k^\vee(m) &= T^m(\alpha_k^\vee) = \alpha_k^\vee + (m_{k+1} - m_k) \delta^\vee, \\ \tau_k(m) &= T^m(\tau_k) = \tau_{(\Lambda_k + m)}. \end{aligned}$$

Here we assume that $m_{k+n} = m_k$ for $k \in \mathbb{Z}$. Then we have

$$\begin{aligned} \Lambda_k &= \Lambda_{k-1} + \varepsilon_k, & \Lambda_{k+1} &= \Lambda_{k-1} + \varepsilon_k + \varepsilon_{k+1}, \\ s_k(\Lambda_k) &= \Lambda_{k-1} + \varepsilon_{k+1}, & s_{k+1} s_k(\Lambda_{k+1}) &= \Lambda_{k-1} + \varepsilon_{k+2}, \\ s_{k+1}(\Lambda_{k+1}) &= \Lambda_{k-1} + \varepsilon_k + \varepsilon_{k+2}, & s_k s_{k+1}(\Lambda_{k+1}) &= \Lambda_{k-1} + \varepsilon_{k+1} + \varepsilon_{k+2}. \end{aligned}$$

Therefore, applying T^m to the both-hand sides of the formula in Lemma 2.3, we obtain

$$\begin{aligned} &[\alpha_{k+1}^\vee(m)]_q \tau_{k-1}(m + \varepsilon_k) \tau_{k-1}(m + \varepsilon_{k+1} + \varepsilon_{k+2}) \\ &+ [\alpha_k^\vee(m)]_q \tau_{k-1}(m + \varepsilon_{k+2}) \tau_{k-1}(m + \varepsilon_k + \varepsilon_{k+1}) \\ &= [\alpha_k^\vee(m) + \alpha_{k+1}^\vee(m)]_q \tau_{k-1}(m + \varepsilon_{k+1}) \tau_{k-1}(m + \varepsilon_k + \varepsilon_{k+2}). \end{aligned}$$

This equation is rewritten in the following cyclically symmetric form:

$$\begin{aligned} &[\varepsilon_{k+1}^\vee(m) - \varepsilon_{k+2}^\vee(m)]_q \tau_{k-1}(m + \varepsilon_k) \tau_{k-1}(m + \varepsilon_{k+1} + \varepsilon_{k+2}) \\ &+ [\varepsilon_k^\vee(m) - \varepsilon_{k+1}^\vee(m)]_q \tau_{k-1}(m + \varepsilon_{k+2}) \tau_{k-1}(m + \varepsilon_k + \varepsilon_{k+1}) \\ &+ [\varepsilon_{k+2}^\vee(m) - \varepsilon_k^\vee(m)]_q \tau_{k-1}(m + \varepsilon_{k+1}) \tau_{k-1}(m + \varepsilon_k + \varepsilon_{k+2}) = 0. \end{aligned} \tag{2.1}$$

We call this equation *the quantum q -Hirota-Miwa equation*. The original (non-quantum) Hirota-Miwa equation is found in Equation (2.1) of [10] and in Equation (2.6) of [19]. Although the method for the above derivation is same as the one in Section 4.5 of [23], we must be careful to the non-commutativity mentioned in Remark 2.4.

2.3 Definition of regularity of the quantum τ -functions

Definition 2.5. For each $\nu \in WP_+$, the quantum τ -function $\tau_{(\nu)}$ is said to be *regular* if $\tau_{(\nu)} \in A^{\text{pa}}$, namely, if $\tau_{(\nu)}$ is a polynomial in $\{f_i, \alpha_i^\vee\}_{i \in I}$ for the Kac-Moody case, and if $\tau_{(\nu)}$ is a polynomial in $\{f_i, q^{\pm \alpha_i^\vee}\}_{i \in I}$ for the q -difference case. \square

As mentioned in the introduction, the regularity of the classical τ -functions is shown by Noumi and Yamada in [25]. The rest of this paper is devoted to the proof of the regularity of the quantum τ -functions.

It is sufficient for the proof of the regularity of the quantum τ -functions for any A to obtain the regularity for $A = U(\mathfrak{n}_-)$ in the Kac-Moody case and for $A = U_q(\mathfrak{n}_-)$ in the q -difference case. Therefore, in the following subsections, we set $A = U_- = U(\mathfrak{n}_-)$, $A^{\text{pa}} = U_-^{\text{pa}} = U_-[\beta | \beta \in Q^\vee]$, $\tilde{A}^{\text{pa}} = \tilde{U}_-^{\text{pa}} = U_-[f_i^{-1}, \beta | i \in I, \beta \in Q^\vee]$, $U = U(\mathfrak{g})$ in the Kac-Moody case, and $A = U_- = U_q(\mathfrak{n}_-)$, $A^{\text{pa}} = U_-^{\text{pa}} = U_-[q^\beta | \beta \in Q^\vee]$, $\tilde{A}^{\text{pa}} = \tilde{U}_-^{\text{pa}} = U_-[f_i^{-1}, q^\beta | i \in I, \beta \in Q^\vee]$, $U = U_q(\mathfrak{g})$ in the q -difference case. Moreover we assume that $\{\alpha_i\}_{i \in I}$ is also linearly independent over \mathbb{Z} .

2.4 Relation to singular vectors in Verma modules

In the following we fix $\mu \in P_+$ and assume that $w_n = s_{i_n} \cdots s_{i_2} s_{i_1}$ is a reduced expressions of $w_n \in W$ for each $n = 0, 1, \dots, N$. We set $\beta_n = w_{n-1}^{-1}(\alpha_{i_n}^\vee) = s_{i_1} s_{i_2} \cdots s_{i_{n-1}}(\alpha_{i_n}^\vee)$ for $n = 1, 2, \dots, N$.

By the definition of the quantum birational Weyl group action (Definition 1.12), for $n = 0, 1, \dots, N$, we have

$$\begin{aligned} \tilde{w}_n^{-1} \tau_{(w_n(\mu))} &= \tilde{w}_n^{-1} \text{Ad}(f_{i_n}^{\alpha_{i_n}^\vee}) \tilde{s}_{i_n} \cdots \text{Ad}(f_{i_2}^{\alpha_{i_2}^\vee}) \tilde{s}_{i_2} \text{Ad}(f_{i_1}^{\alpha_{i_1}^\vee}) \tilde{s}_{i_1}(\tau^\mu) \\ &= \text{Ad}(f_{i_n}^{-\beta_n}) \cdots \text{Ad}(f_{i_2}^{-\beta_2}) \text{Ad}(f_{i_1}^{-\beta_1})(\tau^\mu) \\ &= f_{i_n}^{-\beta_n} \cdots f_{i_2}^{-\beta_2} f_{i_1}^{-\beta_1} \tau^\mu f_{i_1}^{\beta_1} f_{i_2}^{\beta_2} \cdots f_{i_n}^{\beta_n} \\ &= f_{i_n}^{-\beta_n} \cdots f_{i_2}^{-\beta_2} f_{i_1}^{-\beta_1} f_{i_1}^{\beta_1 + \langle \beta_1, \mu \rangle} f_{i_2}^{\beta_2 + \langle \beta_2, \mu \rangle} \cdots f_{i_n}^{\beta_n + \langle \beta_n, \mu \rangle} \tau^\mu. \end{aligned}$$

Thus we obtain

$$\tilde{w}_n^{-1} \tau_{(w_n(\mu))} = \Phi_n^{-1} \Psi_n \tau^\mu,$$

where Φ_n and Ψ_n are given by

$$\Phi_n = f_{i_1}^{\beta_1} f_{i_2}^{\beta_2} \cdots f_{i_n}^{\beta_n}, \quad \Psi_n = f_{i_1}^{\beta_1 + \langle \beta_1, \mu \rangle} f_{i_2}^{\beta_2 + \langle \beta_2, \mu \rangle} \cdots f_{i_n}^{\beta_n + \langle \beta_n, \mu \rangle}.$$

This is equivalent to $\tau_{(w_n(\mu))} = \tilde{w}_n(\Phi_n^{-1} \Psi_n) \tau^{w_n(\mu)}$.

Remark 2.6. In general, for $w \in W$ and $\mu \in P_+$, there exists a unique $\phi_w(\mu) \in \mathcal{A}$ with $\tau_{(w(\mu))} = \phi_w(\mu) \tau^{w(\mu)}$. In [24], the classical version of $\phi_w(\mu)$ is called the τ -cocycle. In the quantum case, $\phi_w(\mu)$ does not commute with τ^μ in general. This is the reason why we does not deal with $\phi_w(\mu)$ but $\tau_{(w(\mu))}$. \square

Let $\sigma : U^- \rightarrow U^-$ be the anti-algebra involution of U_- which sends f_i to f_i ($i \in I$). That is, the linear transformation σ reverses the order of products of f_i 's in U_- . Denote the unique extension of σ to the anti-algebra involution of $\tilde{U}_- = U_-[f_i^{-1} | i \in I]$ by the same symbol.

Assume that $\lambda \in P$. The Verma module $M(\lambda)$ is defined to be the left U -module generated by v_λ with fundamental relations $e_i v_\lambda = 0$ ($i \in I$), $h_i v_\lambda = \langle \alpha_i^\vee, \lambda \rangle v_\lambda$ ($i \in I$) in the Kac-Moody case and $e_i v_\lambda = 0$ ($i \in I$), $q^{h_i} v_\lambda = q^{\langle \alpha_i^\vee, \lambda \rangle} v_\lambda$ ($i \in I$) in the q -difference case. The vector v_λ is called the highest weight vector of $M(\lambda)$.

For $\lambda \in P$, the highest weight simple module $L(\lambda)$ is defined to be the unique simple quotient of the Verma module $M(\lambda)$. The highest weight vector u_λ of $L(\lambda)$ is defined to be the image of $v_\lambda \in M(\lambda)$ in $L(\lambda)$. The simple module $L(\lambda)$ is integrable if and only if $\lambda \in P_+$.

Recall that the Weyl vector $\rho \in P_+$ satisfies $\langle \alpha_i^\vee, \rho \rangle = 1$ ($i \in I$). Define the shifted action of the Weyl group on P by $w \circ \lambda = w(\lambda + \rho) - \rho$ ($w \in W$, $\lambda \in P$).

Assume that $\lambda \in P_+$ and $w \in W$. We define $F_{w,\lambda} \in U_-$ by

$$F_{w,\lambda} = f_{j_m}^{\langle \alpha_{j_m}^\vee, s_{m_l-1} \cdots s_{j_2} s_{j_1} \circ \lambda \rangle + 1} \cdots f_{j_2}^{\langle \alpha_{j_2}^\vee, s_{j_1} \circ \lambda \rangle + 1} f_{j_1}^{\langle \alpha_{j_1}^\vee, \lambda \rangle + 1},$$

where $w = s_{j_m} \cdots s_{j_2} s_{j_1}$ is a reduced expression of w . Note that $F_{w,\lambda}$ is independent on the choice of the reduced expression of w owing to the Verma identities, and $F_{w,\lambda} v_\lambda$ is a singular vector with weight $w \circ \lambda$ in the Verma module $M(\lambda)$, which is unique up to scalar multiples. (For the uniqueness, see 4.4.15 of [13].) For $\lambda \in P_+$, the kernel of the canonical projection from $M(\lambda)$ onto $L(\lambda)$ is generated by $\{F_{s_i, \lambda} v_\lambda = f_i^{\langle \alpha_i^\vee, \lambda \rangle + 1} v_\lambda\}_{i \in I}$.

Using $\langle \beta_k, \lambda + \rho \rangle = \langle \alpha_{i_k}^\vee, s_{i_{k-1}} \cdots s_{i_2} s_{i_1} \circ (\lambda) \rangle + 1$ for $k = 0, 1, \dots, N$, we obtain

$$\sigma(\phi_{\lambda+\rho}(\Phi_n)) = F_{w_n, \lambda}, \quad \sigma(\phi_{\lambda+\rho}(\Psi_n)) = F_{w_n, \lambda+\mu}.$$

The quantum τ -function $\tau_{(w_n(\mu))}$ and the singular vectors $F_{w_n, \lambda} v_\lambda \in M(\lambda)$, $F_{w_n, \lambda+\mu} v_{\lambda+\mu} \in M(\lambda + \mu)$ are related in this way.

For each $i \in I$, the multiplicative subset of U_- generated by the single f_i is an Ore set in U_- owing to the Serre and q -Serre relations. Therefore we obtain the localization $U_-[f_i^{-1}] \subset \tilde{U}_-$ of U_- with respect to it.

For each $i \in I$, we set $U_-[f_i^{-1}]^{\text{pa}} = U_-[f_i^{-1}][\beta | \beta \in Q^\vee]$ in the Kac-Moody case and $U_-[f_i^{-1}]^{\text{pa}} = U_-[f_i^{-1}][q^\beta | \beta \in Q^\vee]$ in the q -difference case. Then $U_-[f_i^{-1}]^{\text{pa}}$ is a subalgebra of \tilde{U}_-^{pa} .

For each $\lambda \in P$, the algebra homomorphism $\phi_\lambda : U_-[f_i^{-1}]^{\text{pa}} \rightarrow U_-[f_i^{-1}]$ is defined to be the restriction of $\phi_\lambda : \tilde{U}_-^{\text{pa}} = \tilde{A}^{\text{pa}} \rightarrow \tilde{A} = \tilde{U}_-$ on $U_-[f_i^{-1}]^{\text{pa}}$.

Lemma 2.7. *For any $a \in U_-[f_i^{-1}]^{\text{pa}}$, if $\phi_{\lambda+\rho}(a) \in U_-$ for all $\lambda \in P_+$, then $a \in U_-^{\text{pa}}$.*

Proof. Let $U_-[i]$ be the subalgebra of U_- generated by $\sigma((\text{ad } f_i)^k(f_j))$ for $j \in I \setminus \{i\}$ and $k = 0, 1, \dots, -a_{ij}$. Then $U_- = \bigoplus_{k=0}^{\infty} U_-[i] f_i^k$ (Section 38.1 of [17]). Therefore we have $U_-[f_i^{-1}] = \bigoplus_{k \in \mathbb{Z}} U_-[i] f_i^k$. Put $U_-[i]^{\text{pa}} = U_-[i][\alpha_j^\vee | j \in I]$ in the Kac-Moody case and $U_-[i]^{\text{pa}} = U_-[i][q^\beta | \beta \in Q^\vee]$ in the q -difference case. Then we have $U_-[f_i^{-1}]^{\text{pa}} = \bigoplus_{k \in \mathbb{Z}} U_-[i]^{\text{pa}} f_i^k$. Therefore any $a \in U_-[f_i^{-1}]^{\text{pa}}$ is uniquely expressed as $a = \sum_{k \in \mathbb{Z}} a_k f_i^k$ where

all $a_k \in U_-[i]^{\text{pa}}$ ($k \in \mathbb{Z}$) are zero except for finite number of k . Then $\phi_\lambda(a) = \sum_{k \in \mathbb{Z}} \phi_\lambda(a_k) f_i^k$. Therefore $\phi_{\lambda+\rho}(a) \in U_-$ for all $\lambda \in P_+$ implies $a_k = 0$ for all $k < 0$. Thus we obtain $a \in U_-^{\text{pa}}$. \square

Lemma 2.8. *If $F_{w_n, \lambda+\mu} \in U_- F_{w_n, \lambda}$ for all $\lambda \in P_+$ and $n = 0, 1, 2, \dots, N$, then the quantum τ -function $\tau_{(w_n(\mu))}$ is regular for each $n = 0, 1, \dots, N$.*

Proof. The quantum τ -function $\tau_{(w_n(\mu))}$ is regular if and only if $\Phi_n^{-1} \Psi_n \in U_-^{\text{pa}}$. Inductively we assume that $1 \leq n \leq N$ and $\Phi_{n-1}^{-1} \Psi_{n-1} \in U_-^{\text{pa}}$. Lemma 1.15 leads to

$$\Phi_n^{-1} \Psi_n = f_{i_n}^{-\beta_n} \Phi_{n-1}^{-1} \Psi_{n-1} f_{i_n}^{\beta_n + \langle \beta_n, \mu \rangle} \in U_-[f_{i_n}^{-1}]^{\text{pa}}.$$

We have $\sigma(\phi_{\lambda+\rho}(\Phi_n^{-1} \Psi_n)) = F_{w_n, \lambda+\mu} F_{w_n, \lambda}^{-1}$. By Lemma 2.7, if $F_{w_n, \lambda+\mu} \in U_- F_{w_n, \lambda}$ for all $\lambda \in P_+$, then $\Phi_n^{-1} \Psi_n \in U_-^{\text{pa}}$. \square

Lemma 2.8 immediately leads to the following proposition.

Proposition 2.9. *If $F_{w, \lambda+\mu} \in U_- F_{w, \lambda}$ for all $\lambda, \mu \in P_+$ and $w \in W$, then the all quantum τ -functions $\tau_{(w(\mu))}$ are regular.* \square

In this way, we can reduce the regularity of the quantum τ -functions to the divisibility (from the right) of $F_{w, \lambda+\mu}$ by $F_{w, \lambda}$ for $\lambda, \mu \in P_+$ and $w \in W$.

2.5 Proof of regularity in the Kac-Moody case

In this subsection, we assume that $A = U_- = U(\mathfrak{n}_-)$ and shall use the integral weight part (P -part) of the results of [2] on the BGG category for the Kac-Moody algebra \mathfrak{g} .

A left \mathfrak{g} -module M is said to be integrally \mathfrak{h} -diagonalizable if $M = \bigoplus_{\nu \in P} M_\nu$ where $M_\nu = \{v \in M \mid h_i v = \langle \alpha_i^\vee, \nu \rangle v \text{ } (i \in I)\}$ for $\nu \in P$. We call M_ν 's the weight subspaces of M and ν a weight of M if $M_\nu \neq 0$.

Let \mathcal{O}_P be the category of left \mathfrak{g} -modules M satisfying the following conditions:

- (A) M is integrally \mathfrak{h} -diagonalizable with finite-dimensional weight spaces;
- (B) There exists finitely many weights $\mu_1, \dots, \mu_n \in P$ such that any weight of M belongs to $\bigcup_{k=1}^n (\mu_k - Q_+)$.

Then we have the Verma module $M(\lambda)$ and its simple quotient $L(\lambda)$ are objects of \mathcal{O}_P if $\lambda \in P$. The simple module $L(\lambda)$ is integrable if and only if $\lambda \in P_+$.

Define the subset $K_P^{\mathfrak{g}}$ of the weight lattice P by

$$K_P^{\mathfrak{g}} = W \circ P_+ = W(P_+ + \rho) - \rho = \{w \circ \lambda = w(\lambda + \rho) - \rho \mid w \in W, \lambda \in P_+\}.$$

Let $\mathcal{O}_P^{\mathfrak{g}}$ be the full subcategory of \mathcal{O}_P consisting of \mathcal{O}_P -objects all simple subquotients of which are isomorphic to $L(\nu)$ for some $\nu \in K_P^{\mathfrak{g}}$. Then the Verma modules $M(w \circ \lambda)$ for $w \in W$ and $\lambda \in P_+$ are objects of $\mathcal{O}_P^{\mathfrak{g}}$.

For $\lambda \in P_+$, let \mathcal{O}_λ be the full subcategory of $\mathcal{O}_P^{\mathfrak{g}}$ consisting of $\mathcal{O}_P^{\mathfrak{g}}$ -objects all simple subquotients of which are isomorphic to $L(w \circ \lambda)$ for some $w \in W$. Then any $\mathcal{O}_P^{\mathfrak{g}}$ -object

decomposes uniquely as a direct sum of \mathcal{O}_λ -objects for $\lambda \in P_+$ (the integral part of Theorem 5.7 of [2]). For any $\mathcal{O}_P^\mathfrak{g}$ -object M , denote by $\text{pr}_\lambda(M)$ the \mathcal{O}_λ -component of M .

For $\mu, \lambda \in P_+$, the translation functor $T_\mu^{\lambda+\mu} : \mathcal{O}_\lambda \rightarrow \mathcal{O}_{\lambda+\mu}$ is defined by $T_{\lambda+\mu}^\lambda(M) = \text{pr}_{\lambda+\mu}(M \otimes L(\mu))$ for $M \in \text{Ob } \mathcal{O}_\lambda$. This functor is exact. The following lemma is the integral part of Theorem 5.13 of [2].

Lemma 2.10. *For $\lambda, \mu \in P_+$ and $w \in W$, the \mathfrak{g} -module $T_\lambda^{\lambda+\mu}(M(w \circ \lambda))$ is isomorphic to $M(w \circ (\lambda + \mu))$. \square*

For details of the theory of translation functor for the Kac-Moody Lie algebras, see [2]. (See also Section 2 of [14].)

The following theorem is a canonically quantized version of Theorem 1.3 of [25].

Theorem 2.11. *In the Kac-Moody case, the all quantum τ -functions $\tau_{(\nu)}$ for $\nu \in WP_+$ are regular. More precisely they are polynomials in $\{f_i, \alpha_i^\vee\}_{i \in I}$.*

Proof. By Proposition 2.9 it is sufficient to show $F_{w, \lambda+\mu} \in U_- F_{w, \lambda}$ for $\lambda, \mu \in P_+$ and $w \in W$. Fix $\lambda, \mu \in P_+$ and $w \in W$. Denote $T_\lambda^{\lambda+\mu}$ by T .

Since $F_{w, \lambda} v_\lambda$ is a singular vector with weight $w \circ \lambda$, we can identify $M(w \circ \lambda)$ with $U_- F_{w, \lambda} v_\lambda \subset M(\lambda)$. We can also identify $M(\lambda + \mu)$ with $U_-(v_\lambda \otimes u_\mu) \subset M(\lambda) \otimes L(\mu)$, and $M(w \circ (\lambda + \mu))$ with $\subset M(\lambda + \mu)$.

By the exactness of the translation functor, we can regard $T(M(w \circ \lambda))$ as a submodule of $T(M(\lambda))$. By Lemma 2.10 and the uniqueness (up to scalar multiples) of the non-zero homomorphism from $M(w \circ (\lambda + \mu))$ to $M(\lambda + \mu)$, we obtain $T(M(w \circ \lambda)) = M(w \circ (\lambda + \mu)) = U_- F_{w, \lambda+\mu}(v_\lambda \otimes u_\mu)$. Therefore

$$F_{w, \lambda+\mu}(v_\lambda \otimes u_\mu) \in M(w \circ (\lambda + \mu)) = T(M(w \circ \lambda)) \subset M(w \circ \lambda) \otimes L(\mu).$$

On the other hand, we can rewrite $F_{w, \lambda+\mu}(v_\lambda \otimes u_\mu)$ in the following form:

$$F_{w, \lambda+\mu}(v_\lambda \otimes u_\mu) = (F_{w, \lambda+\mu} v_\lambda) \otimes u_\mu + \sum_k a_k \otimes u_k,$$

where $\{u_k\}$ is a basis of $\bigoplus_{\nu \neq \mu} L(\mu)_\nu$ and a_k 's are some elements of $M(\lambda)$. Therefore $F_{w, \lambda+\mu} v_\lambda$ and a_k 's belong to $M(w \circ \lambda) = U_- F_{w, \lambda} v_\lambda$. In particular, $F_{w, \lambda+\mu} \in U_- F_{w, \lambda}$. \square

Remark 2.12. The classical limit of Theorem 2.11 gives another proof of the regularity of the classical τ -functions (Theorem 1.3 of [25]). In [25], Noumi and Yamada proves the regularity of the classical τ -functions by using the idea of the Sato theory of soliton equations [30]. Our method is completely different from it. \square

2.6 Proof of regularity in the q -difference case

In this subsection, we fix an arbitrary complex number \hbar which is not a root of unity. In order to prove the regularity of the quantum τ -functions in the q -difference case, it is enough to show the regularity in the case where q is specialized at e^\hbar .

Let $U_h(\mathfrak{g})$ be the associative algebra over \mathbb{C} generated by $\{e_i, f_i, q^\beta \mid i \in I, \beta \in Q^\vee\}$ with the same fundamental relations of $U_q(\mathfrak{g})$ specialized at $q = e^h$. For each $\lambda \in P$, the Verma module $M_h(\lambda)$ over $U_h(\mathfrak{g})$ and its simple quotient $L_h(\lambda)$ are similarly defined as in the case of $U_q(\mathfrak{g})$. The simple module $L_h(\lambda)$ is integrable if and only if $\lambda \in P_+$. For $\lambda \in P_+$ and $w \in W$, there exists a non-zero $U_h(\mathfrak{g})$ -homomorphism from $M_h(w \circ \lambda)$ to $M_h(\lambda)$, which is injective and unique up to scalar multiples. For the uniqueness, see 4.4.15 of [13]. Although it deal with the case where q is an indeterminate, its proof holds for $q = e^h \in \mathbb{C}^\times$ not a root of unity (Section 2.4 of [9]). We regard $M_h(w \circ \lambda)$ as a submodule of $M_h(\lambda)$.

In the following, we assume that $\lambda, \mu \in P_+$ and $w \in W$.

Since the analogue of Proposition 2.9 for $U_h(\mathfrak{g})$ also holds, the regularity of the quantum τ -functions in the q -difference case specialized at $q = e^h$ can be also obtained by the same argument as in Section 2.5 if there exists $U_h(\mathfrak{g})$ -modules $T_h(\lambda)$ and $T_h(w \circ \lambda)$ satisfying the following conditions:

- (a) $T_h(\lambda)$ is a submodule of $M_h(\lambda) \otimes L_h(\mu)$ and isomorphic to $M_h(\lambda + \mu)$.
- (b) $T_h(w \circ \lambda)$ is a submodule of $M_h(w \circ \lambda) \otimes L_h(\mu)$ and isomorphic to $M_h(w \circ (\lambda + \mu))$.
- (c) $T_h(w \circ \lambda)$ is a submodule of $T_h(\lambda)$.

Under the conditions above, replacing $T(M(\lambda))$ and $T(M(w \circ \lambda))$ in the proof of Theorem 2.11 with $T_h(\lambda)$ and $T_h(w \circ \lambda)$ respectively, we obtain the regularity of the q -difference version of the quantum τ -function $\tau_{(w(\mu))} = w(\tau^\mu)$ specialized at $q = e^h$.

A left $U_h(\mathfrak{g})$ -module M is said to be integrally \mathfrak{h} -diagonalizable if $M = \bigoplus_{\nu \in P} M_\nu$ where the weight subspaces are defined by $M_\nu = \{v \in M \mid q^\beta v = e^{h\langle \beta, \nu \rangle} v \ (\beta \in Q^\vee)\}$ for $\nu \in P$. Let $\mathcal{O}_{h,P}$ be the category of left $U_h(\mathfrak{g})$ -modules M satisfying the conditions (A) and (B) in Section 2.5. Then, for each $\lambda \in P$, the Verma module $M_h(\lambda)$ over $U_h(\mathfrak{g})$ and its simple quotient $L_h(\lambda)$ are objects of $\mathcal{O}_{h,P}$.

Let $\{g_k\}$ and $\{g^k\}$ be the dual bases of the symmetrizable Kac-Moody algebra \mathfrak{g} with respect to the canonical non-degenerate symmetric invariant bilinear form on \mathfrak{g} . Set $\Omega = \sum_k g_k \otimes g^k$. Following Drinfeld [4], using the associator, we can define on \mathcal{O}_P the structure of a braided tensor category with braiding $e^{h\Omega}$. (For details, see [4] and [5].) Since h is not a root of unity, the universal R -matrix \mathcal{R} for $U_h(\mathfrak{g})$ is well-defined and the category $\mathcal{O}_{h,P}$ is also a braided tensor category with braiding defined by \mathcal{R} .

By Theorem 4.10 of [6], there exists a braided tensor functor $F_h : \mathcal{O}_P \rightarrow \mathcal{O}_{h,P}$, which is the identity functor at the level of P -graded vector spaces and preserves the Verma modules and the integrable simple modules. In particular, we have

$$\begin{aligned} F_h(M(\lambda)) &= M_h(\lambda), & F_h(M(w \circ \lambda)) &= M_h(w \circ \lambda), & F_h(L(\mu)) &= L_h(\mu), \\ F_h(M(\lambda) \otimes L(\mu)) &= M_h(\lambda) \otimes L_h(\mu), & F_h(M(w \circ \lambda) \otimes L(\mu)) &= M_h(w \circ \lambda) \otimes L_h(\mu), \\ F_h(M(\lambda + \mu)) &= M_h(\lambda + \mu), & F_h(M(w \circ (\lambda + \mu))) &= M_h(w \circ (\lambda + \mu)). \end{aligned}$$

Define the $U_h(\mathfrak{g})$ -modules $T_h(\lambda)$ and $T_h(w \circ \lambda)$ by

$$T_h(\lambda) = F_h(T_\lambda^{\lambda+\mu}(M(\lambda))), \quad T_h(w \circ \lambda) = F_h(T_\lambda^{\lambda+\mu}(M(w \circ \lambda))).$$

Then the conditions (a), (b), and (c) are satisfied. It concludes the following theorem.

Theorem 2.13. *Also in the q -difference case, the all quantum τ -functions $\tau_{(\nu)}$ ($\nu \in WP_+$) are regular. More precisely they are polynomials in $\{f_i, q^{\pm\alpha_i^\vee}\}_{i \in I}$. \square*

This theorem is not only a canonically quantized version of Theorem 1.3 of [25] but also its q -difference analogue.

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